Dynamic Matching Algorithms in Practice

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- New fully dynamic matching algorithms have been developed recently
- Bridge the gap between theory and practice by testing out and comparing these algorithms

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Definition (Matching)

A set of edges $\mathcal{M} \subseteq E$ such that for all pairs of edges $((u, v), (r, s)) \in \mathcal{M} : r, s, u, v$ are distinct.

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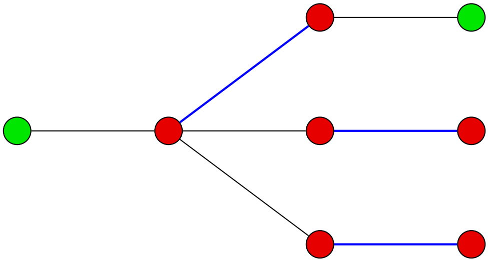
- A matching is called *maximal*, if there is no edge in E that can be added to $\mathcal{M}$



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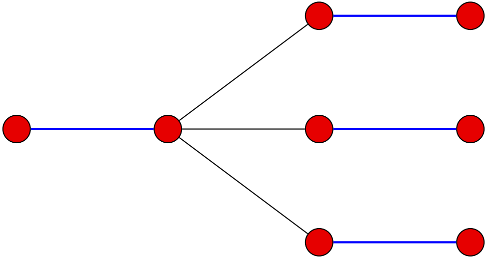
- A *maximum matching* $\mathcal{M}_{\text{opt}}$ is a maximal matching that contains the largest number of possible edges



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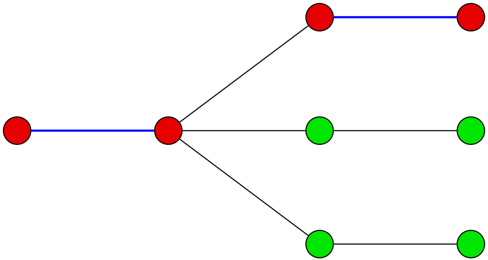
- An α-approximate maximum matching is a matching that contains at least $\frac{|\mathcal{M}_{\text{opt}}|}{\alpha}$ edges



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An augmenting path is a cycle-free path in G that starts and ends on a free vertex and where edges alternate from $\mathcal{M}$ with edges from $E \setminus \mathcal{M}$

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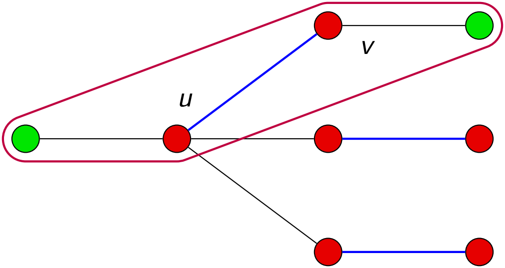

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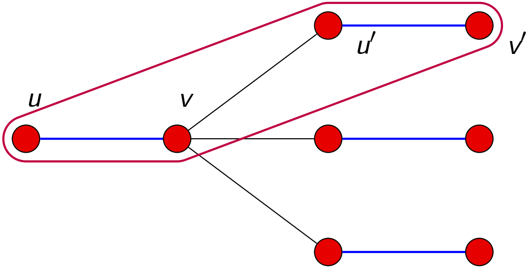

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- In the following, Δ denotes the largest degree that can be found in any state of the dynamic graph

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1. Random Walk-based algorithm:
   - Maintains $(1+\epsilon)$-approximate maximum matching w.h.p.
   - Performs random walks trying to find augmenting paths
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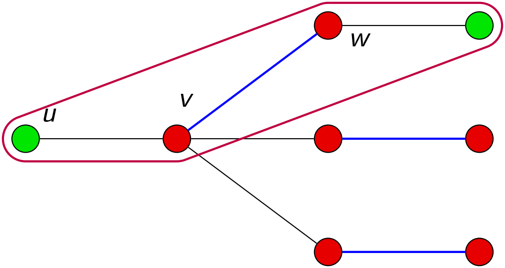
4. Neiman & Solomon (deterministic):
   - Maintains $(\frac{3}{2})$-approximate maximum matching
   - Uses concept of high degree/low degree vertices
   - Update time: $O(\sqrt(m))$ (worst case)

Random Walk-based Algorithm

Random Walk-based Algorithm

1. Pick a free vertex u
2. Randomly choose neighbour v of u
3. If v is free: match (u, v) and stop walk
4. Else: unmatch $(v, mate(v))$ and match (u, v)



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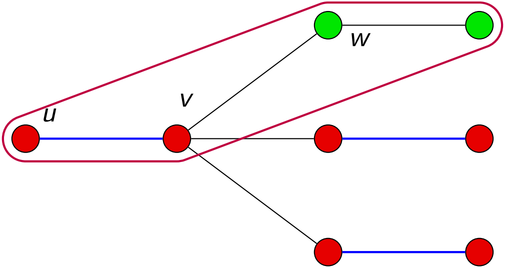
- Now *mate*(v) is free, continue walk from there until $O(\frac{1}{\epsilon})$ steps
- Length of the walk is an important parameter
- Update time for a single walk: $O(\frac{1}{\epsilon})$



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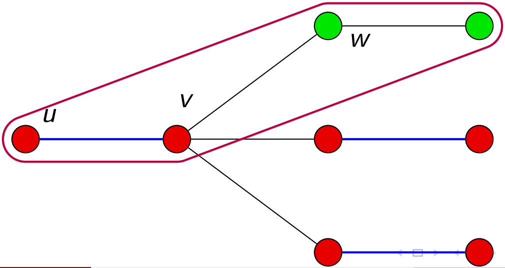
- By itself this does not even guarantee a maximal matching!
- Fixing by undoing all changes
- Alternative: Δ-Settling: Scan through neighbours of visited vertices to find a free vertex
- Stops if either free vertex found or after $\frac{1}{\epsilon}$ steps
- If Random Walk was unsuccessful, try to match the last vertex touched by scanning all its neighbours
- Requires $O(\Delta)$ additional time per visited vertex



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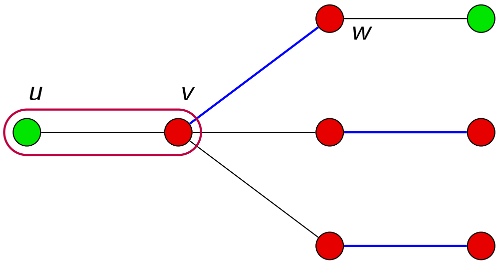

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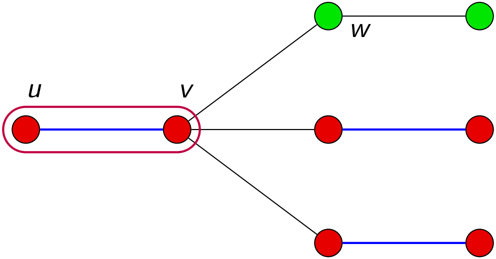

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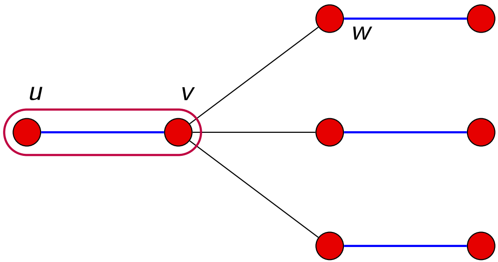

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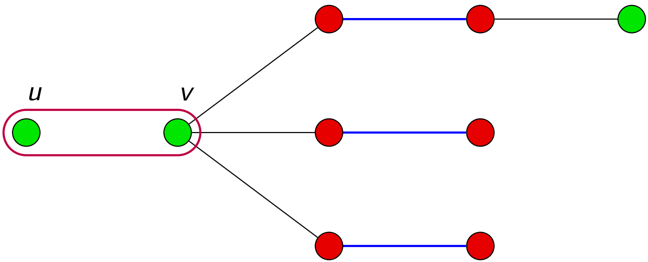

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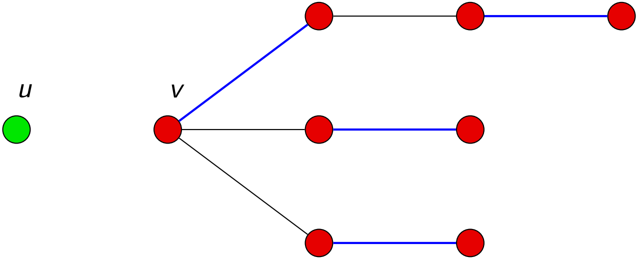

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6. Hence if $\lambda \geq \Delta^{\frac{2}{\epsilon}-1}\log(n)$:

$$(1-\frac{1}{\Delta^{\frac{2}{\epsilon}-1}})^{\Delta^{\frac{2}{\epsilon}-1}\log(n)} \leq e^{-\frac{1}{\Delta^{\frac{2}{\epsilon}-1}}\Delta^{\frac{2}{\epsilon}-1}\log(n)} = e^{-\log(n)} = \frac{1}{n}$$

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- One search stops as soon as augmenting path was found and has running time $\Theta(n' + m')$ where n', m' are the number of vertices and edges touched by the BFS
- First BFS needs an additional $O(n + m)$ time to initialize the data structures, all others do book keeping of the changes they made and undo them afterwards

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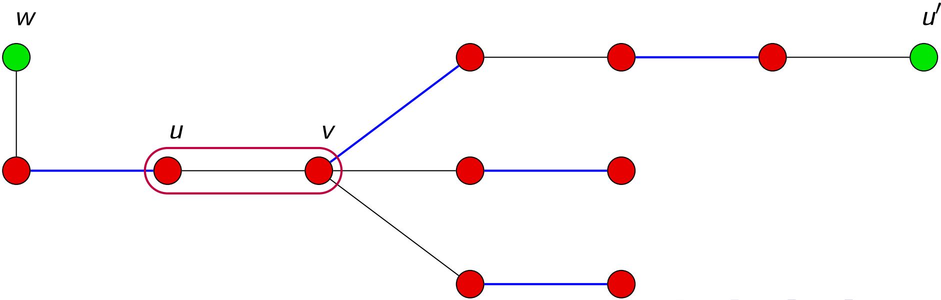

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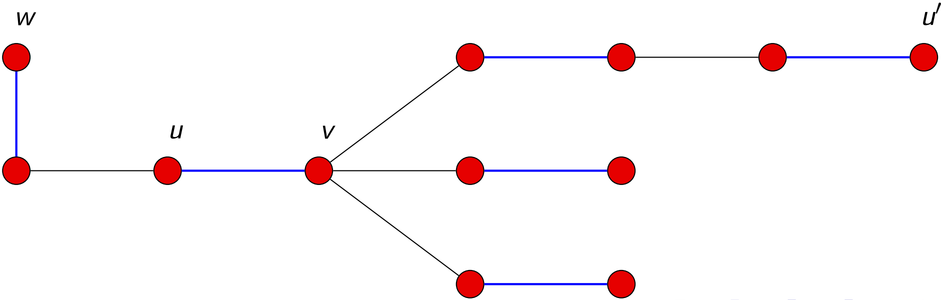

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3. Depth-binding paths to length $\frac{2}{\epsilon} - 1$ ensures deterministic $(1+\epsilon)$-approximate matching algorithm
4. Worst case complexity of optimum version: $O(n+m)$, bounded version: $O(\Delta^{\frac{2}{\epsilon}-1})$
5. Edge Deletions: Start BFS from any free endpoint u or v, combinable with LP and DB

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2. Example graphs include static graphs as well as dynamic ones
3. Two types of experiments: start with empty (static graph) and do insertions only as well as real dynamic graphs
4. Most of the dynamic graph instances only use insertions, deletions are constructed by undoing insertions

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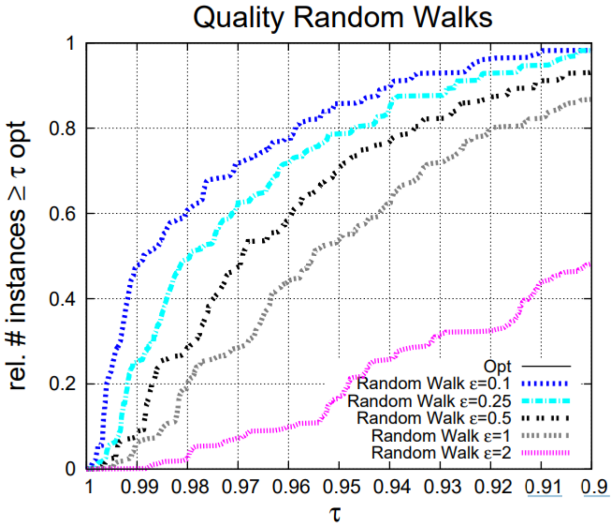

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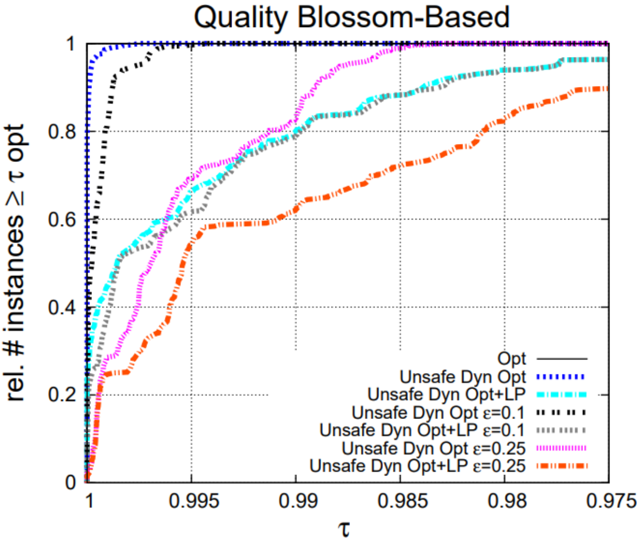

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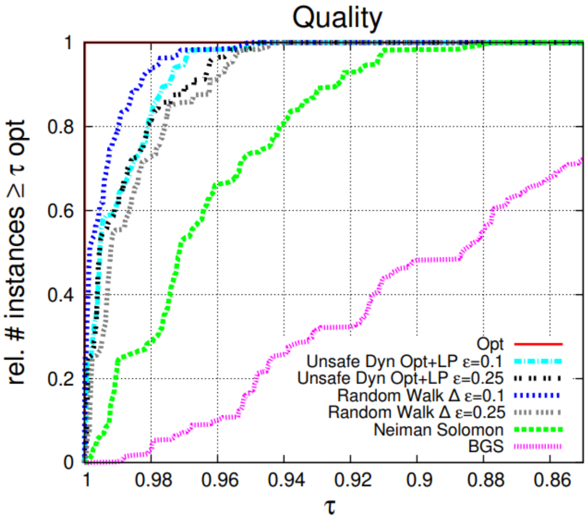

Experiments: Runtime comparison of all algorithms

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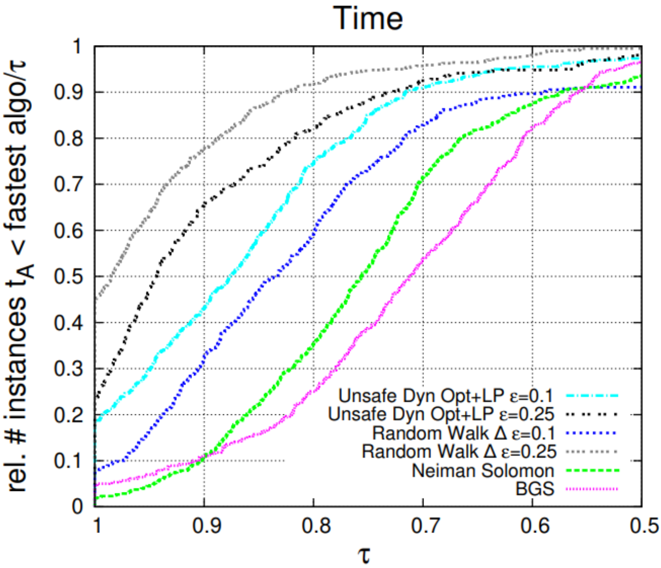

Conclusion

1. Maintaining optimum matchings can be done much more efficiently than the naive approach to compute matchings from scratch after every dynamic change in an unweighted graph
2. All approximative algorithms that we have seen are able to maintain near-optimum matchings in practice while being significantly faster
3. Random-Walks with Delta Settling enabled will be the method of choice in practice
4. Open questions: Weighted case, dynamic multilevel algorithms, parallelization potential, real world dynamic graph instances with both insertions and deletions

Thank you for your attention!