Simple dynamic algorithms for Maximal Independent Set, Maximum Flow and Maximum Matching

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Reading Group Algorithms

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Outline of the paper

- Maximal Independent Set (MIS)
  - Fully dynamic under edge updates
  - $O(\min\{\Delta, m^{\frac{2}{3}}\})$ amortized update time
- Unit-Capacity Maximum Flow (UMF)
  - Incremental under edge updates
  - $O(n)$ amortized update time
- Maximum Cardinality Matching (MCM)
  - Incremental under edge updates
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Maximal Independent Set (MIS)

Definition of MIS

$G = (V, E)$. A subset $M \subseteq V$ is called a *maximal independent set* if and only if:

1. $\forall u, w \in M$: no edge between them
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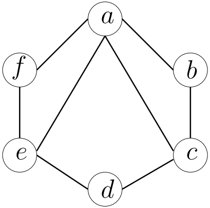

$M = \emptyset$

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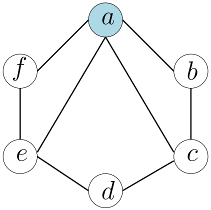

$M = \{a\}$

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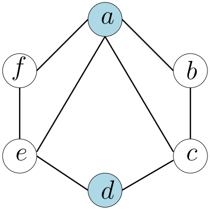

$M = \{a, d\}$

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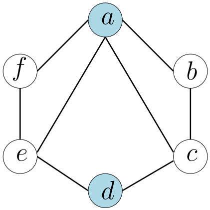

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DFS/BFS: $O(n + m)$

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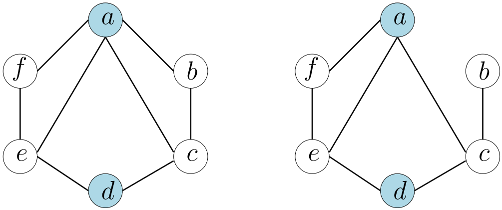

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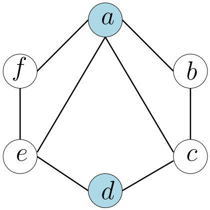

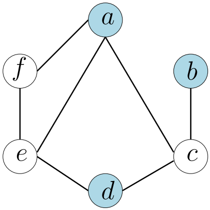

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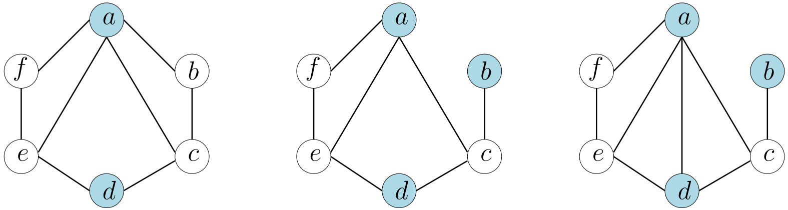

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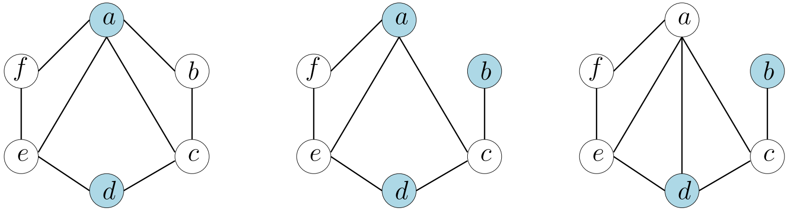

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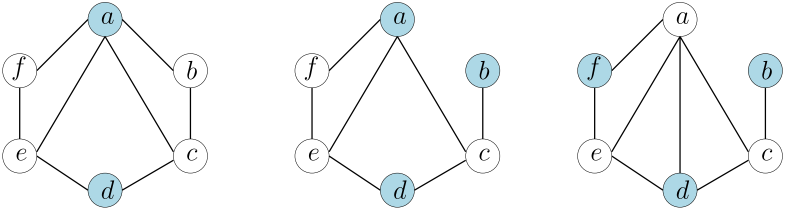
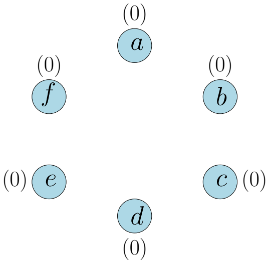


Algorithm with counters



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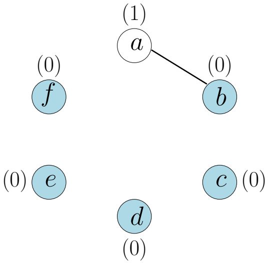

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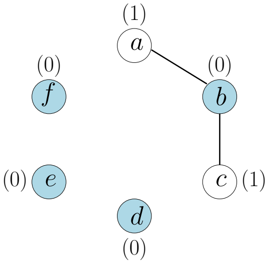

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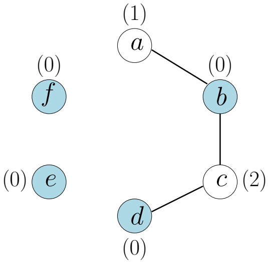

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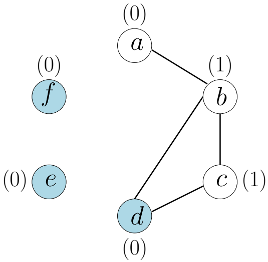

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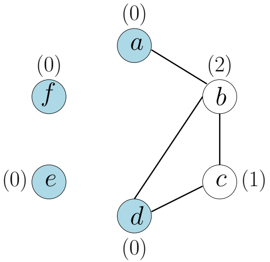

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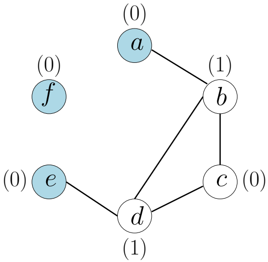

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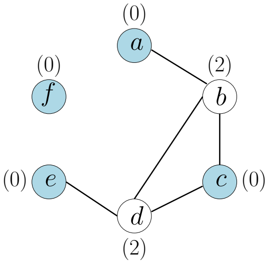

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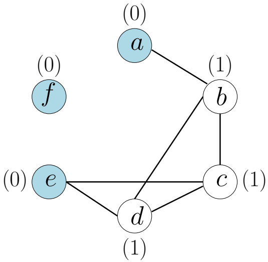

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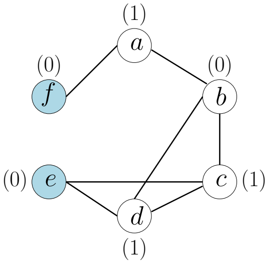

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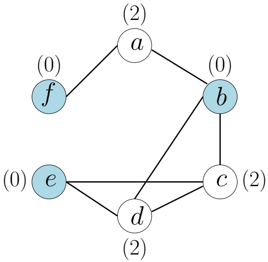

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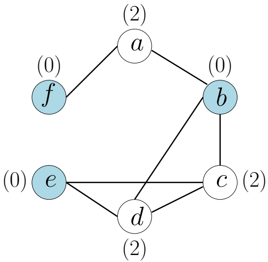
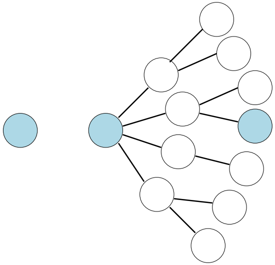


Edge deletion: one vertex may be added to MIS

Edge insertion: one vertex may be removed from MIS

multiple vertices may be added to MIS

Difficult Case



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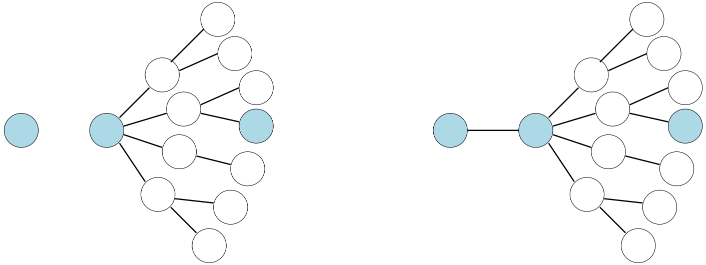

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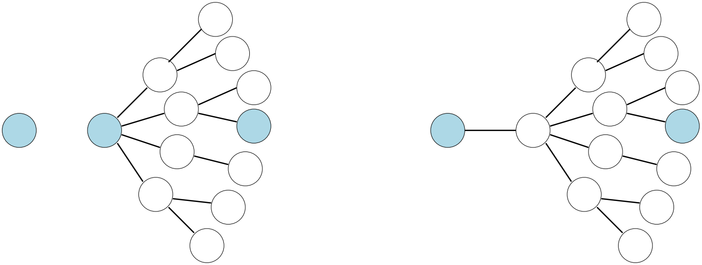

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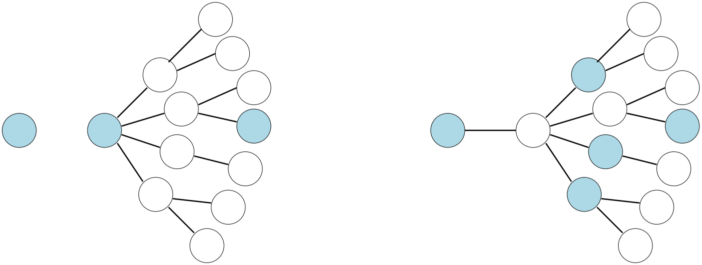

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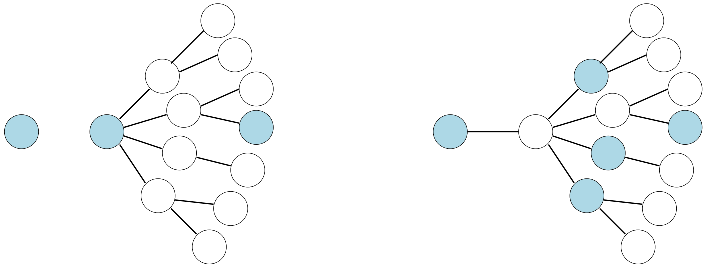
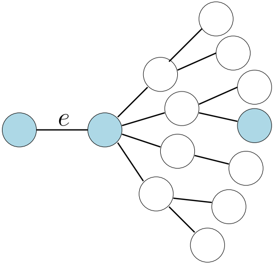


$O(\Delta^2)$ worst-case time

$O(\Delta)$ amortized update time



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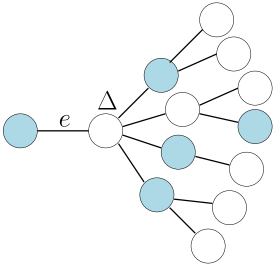

1. At most one vertex may be removed from MIS
2. A vertex iterates over its neighborhood only if enters/leaves MIS

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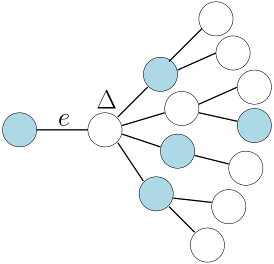

e_1	e_2	e_3	e_4	e_5
-a	-b	-c	-a	$+a$
		$+a$	$+c$	
		+b		

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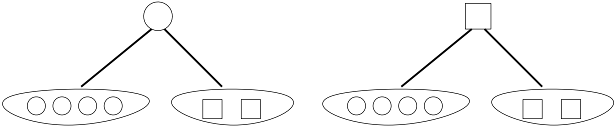
Heavy/Light separation

$$L = \{u \in V(G) \mid \deg(u) \leq \alpha\}$$
$$H = \{u \in V(G) \mid \deg(u) > \alpha\}$$



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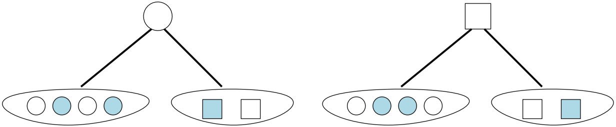

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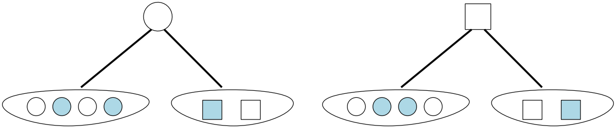

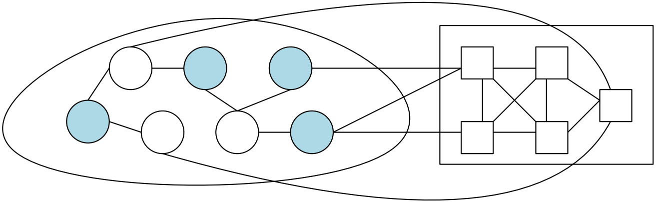

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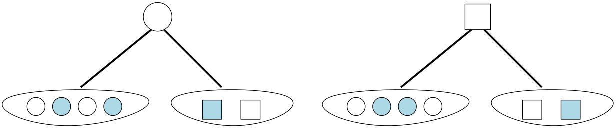

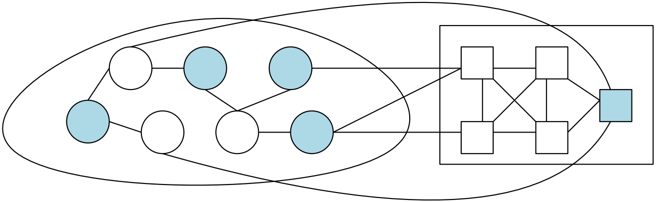

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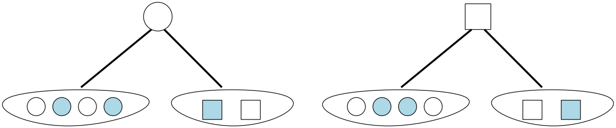

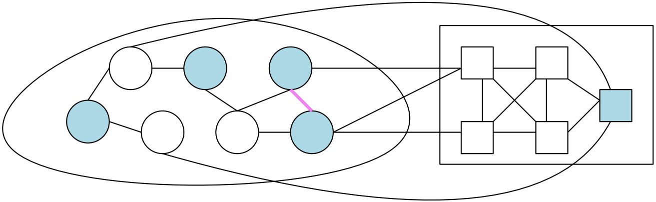

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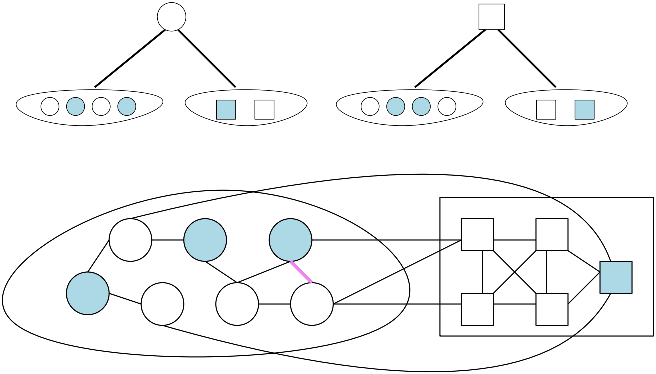

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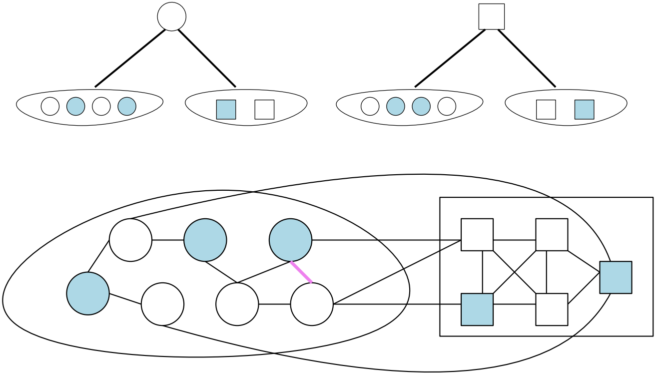

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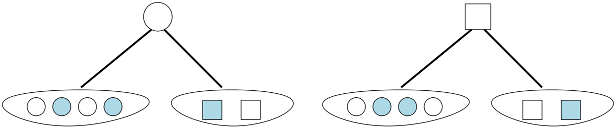

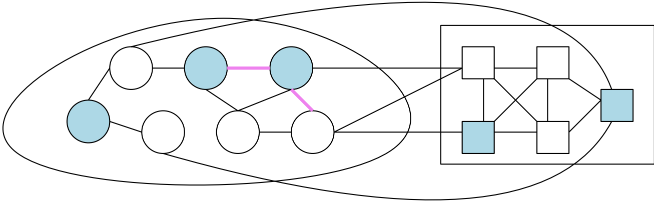

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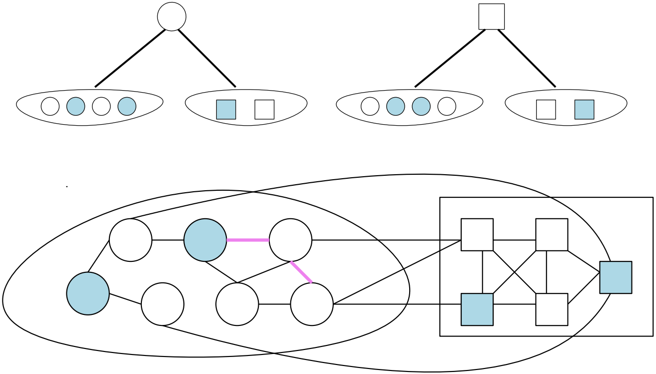

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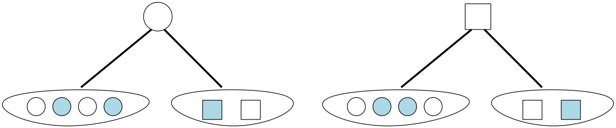

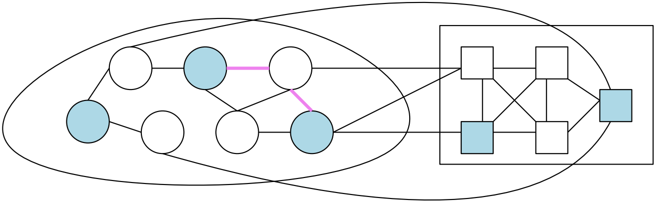

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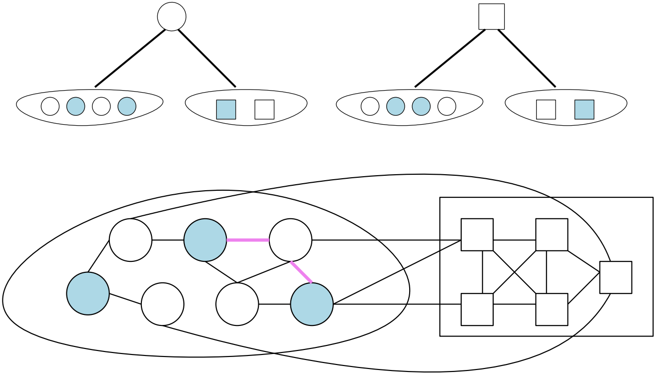

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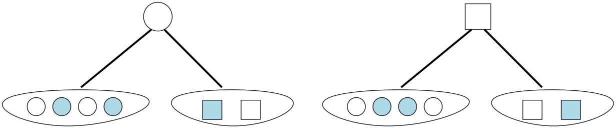

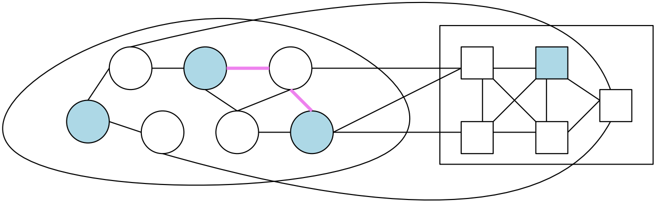

Analyzing Time Complexity

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Deterministic Algorithm:

$O(\min\{\Delta, m^{\frac{2}{3}}\})$ amortized update time

$O(1)$ worst-case query time

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Why is this algorithm faster?

Analyzing Time Complexity

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Heavy vertices do not inform their light neighborhood

Summary of the algorithm

- Split the vertices into two groups: light and heavy

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- ▷ Second step: Apply the $O(m)$ algorithm for the heavy vertices with no light neighbor in MIS

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- ▷ First step: Apply the $O(\Delta)$ algorithm for the light vertices
- ▷ Second step: Apply the $O(m)$ algorithm for the heavy vertices with no light neighbor in MIS

Thank you !

[1] Assadi et. al. "Fully dynamic maximal independent set with sublinear in n update time". **in** 2019.

[2] Manoj Gupta **and** Shahbaz Khan. "Simple dynamic algorithms for Maximal Independent Set, Maximum Flow and Maximum Matching". **in** 2021.