Distributed Approximate Single-Source Shortest Paths

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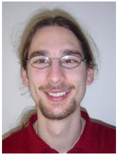
joint works with

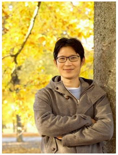
Ruben Becker

Monika Henzinger

Andreas Karrenbauer



Christoph Lenzen



Danupon Nanongkai

One Problem – Two Results

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Comparison:

- Lower bound: $\tilde{\Omega}(\sqrt{n} + \mathit{Diam})$ rounds [Das Sarma et al '11]
- Exact SSSP: $O((n \log n)^{2/3} \mathit{Diam}^{1/3})$ rounds (randomized) [Elkin '17]
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Today: Weighted undirected graphs

Tight and Tighter

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Combinatorics & Optimization

Model and Problem Statement

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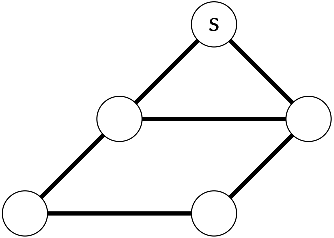
Problem statement:

- Initially, each node knows whether it is the source or not
- Finally: Every node knows its approximate distance to the source
  Often also: Implicit tree; every node knows next edge on approximate shortest path to source

Unweighted Graphs: BFS



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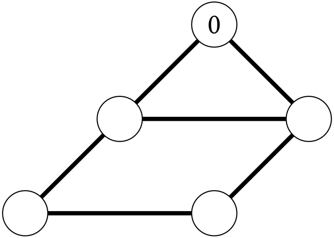

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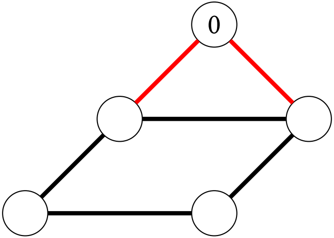

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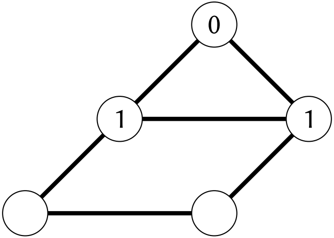

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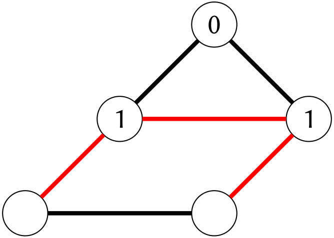

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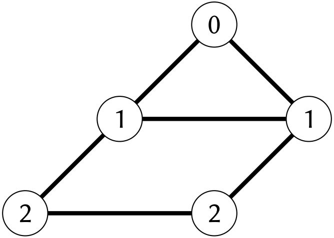

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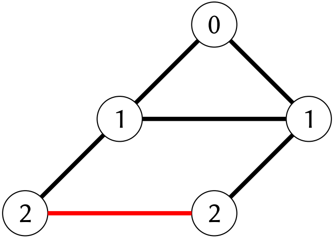

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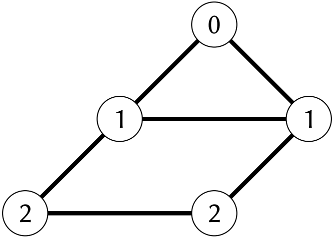

BFS tree can be computed in $O(Diam)$ rounds

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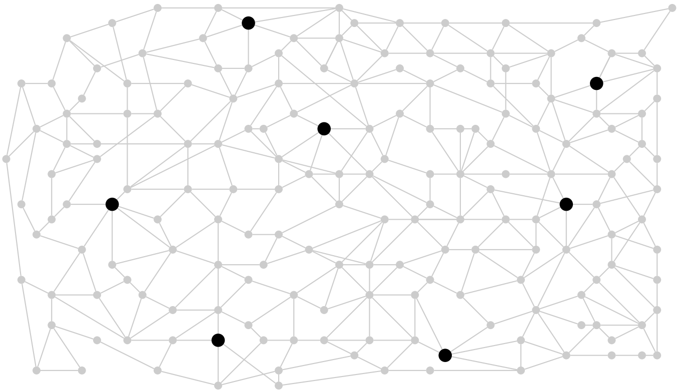

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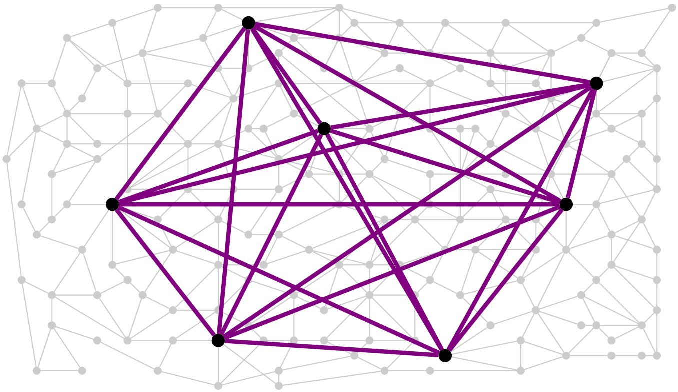

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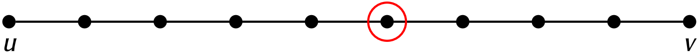
Sample $N = \tilde{O}(n^{1/2})$ centers (+ source s)
$\Rightarrow$ Every shortest path with $\geq n^{1/2}$ edges contains center whp

Derandomization of Overlay Network [HKN '16]

Randomization: Hit every shortest path with $\geq \sqrt{n}$ edges



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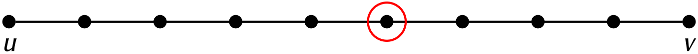
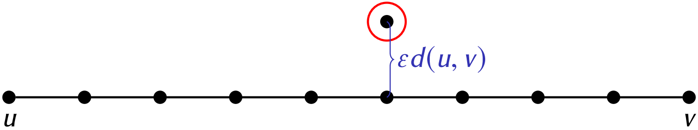
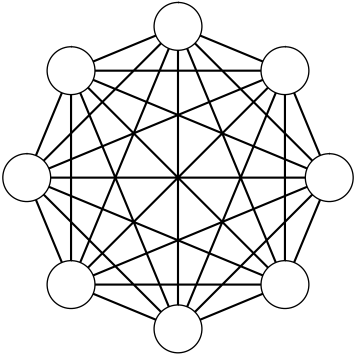


Deterministic relaxation: Almost hit every path $\geq \sqrt{n}$ edges



Congested Clique

Special model: Communication not restricted to neighbors



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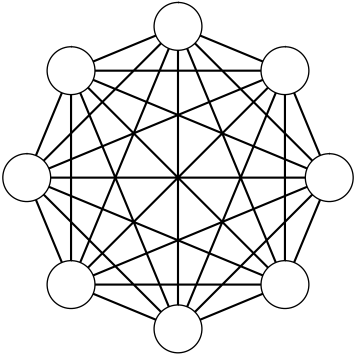

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Simulation: Overlay network as congested clique

t rounds in Congested Clique $\rightarrow \tilde{O}(t \cdot (\sqrt{n} + Diam))$ rounds in CONGEST

Hop Reduction

Well Known: Spanners

Definition

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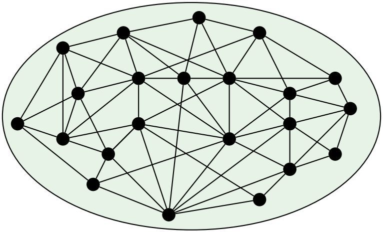

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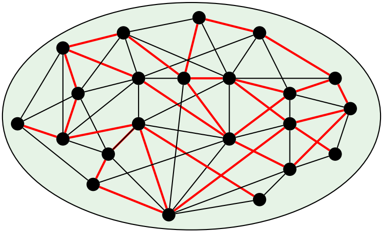

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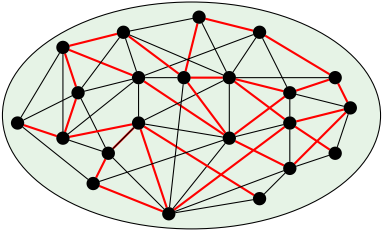

Fact: Every graph has a k-spanner of size $n^{1+1/k}$ [Folklore]

Application: Running time $T(m, n) \Rightarrow T(n^{1+1/k}, n)$

Less Known: Hop Sets

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An (h, ε)-hop set is a set of weighted edges F such that, for all pairs of nodes u and v, in the 'shortcut graph' $G \cup F$ there is a path from u to v with **at most h edges** of weight at most $(1 + \varepsilon)dist(u, v)$.

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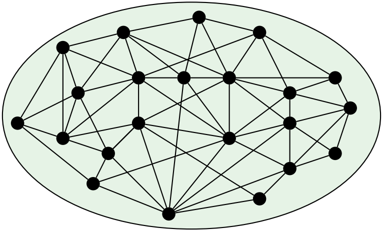

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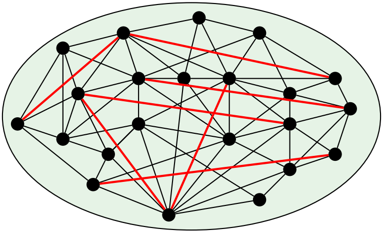

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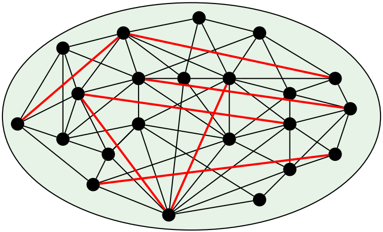

Fact: Every graph has a $(n^{o(1)}, \varepsilon)$-hop set of size $n^{1+o(1)}$ [Cohen '94] (for $\varepsilon \geq 1/polylogn$)

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Remaining challenge: Compute hop set efficiently

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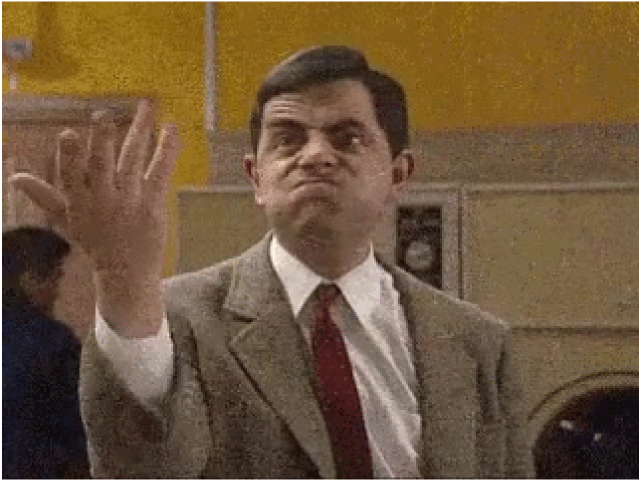
$\Rightarrow$ Cannot have $\alpha = 1+\varepsilon$, $h = poly(1/\varepsilon)$ and size $n \cdot polylog(n)$.



It was too good to be true…

Hop Set Example

Simple Hop Set Based on Balls (following [Thorup/Zwick '06])

$V = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_k = \emptyset$ where node of A_i goes to A_{i+1} with probability $1/n^{1/k}$

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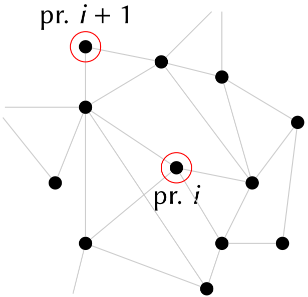

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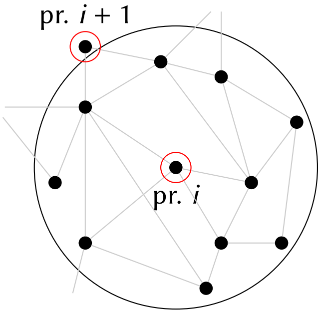

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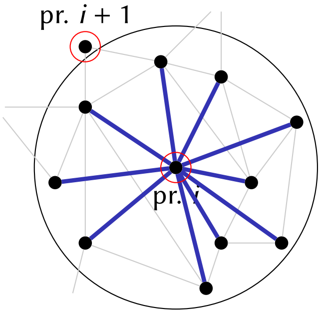

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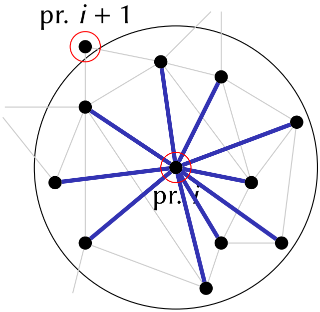

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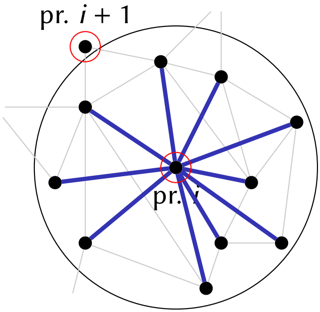

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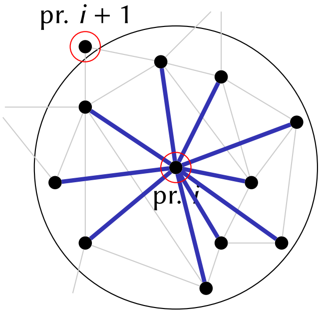

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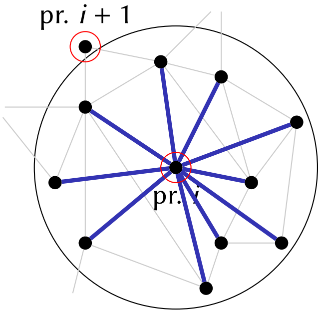

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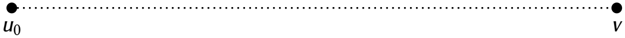
$(n^{1/2+o(1)}, \varepsilon)$-hop set

Case 1: $dist(u_0, v) \leq n^{1/2+1/k}/\varepsilon$



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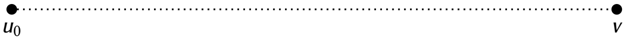

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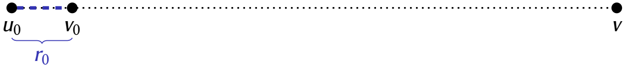

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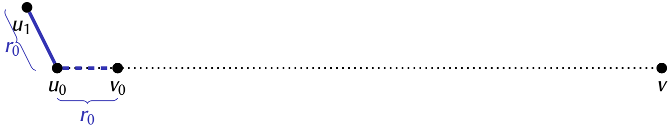

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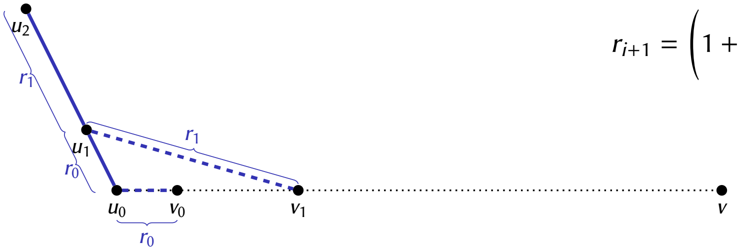

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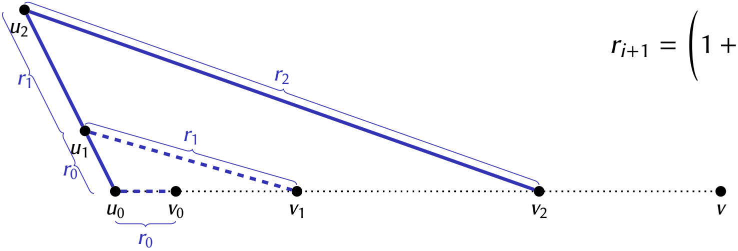

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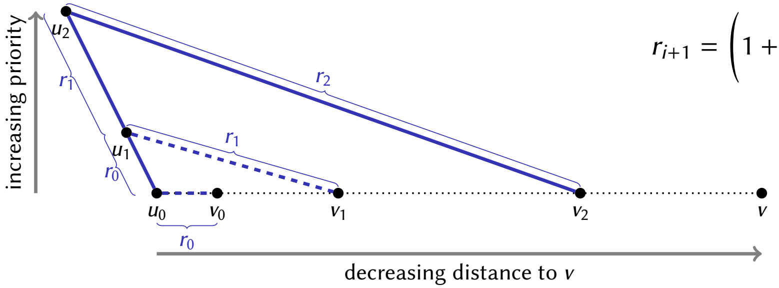

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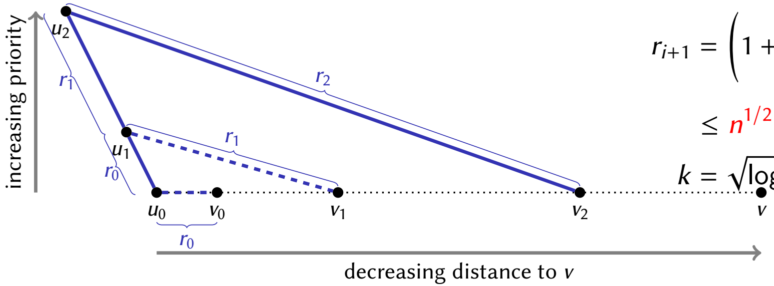

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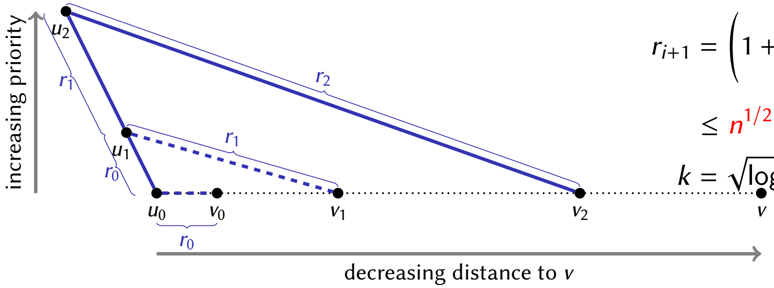

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$$\#\mathit{Edges} \le \frac{k \cdot dist(u, v)}{n^{1/2}} \le \frac{k \cdot n}{n^{1/2}} = kn^{1/2}$$

Chicken-Egg Problem?

1. Goal: Faster SSSP via hop set
2. Compute hop set by computing balls
3. Computing balls at least as hard as SSSP

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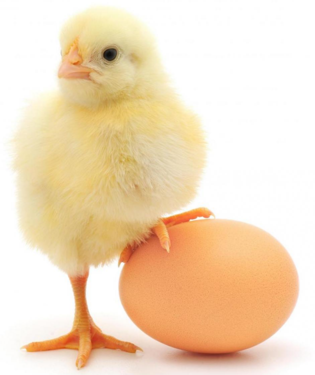

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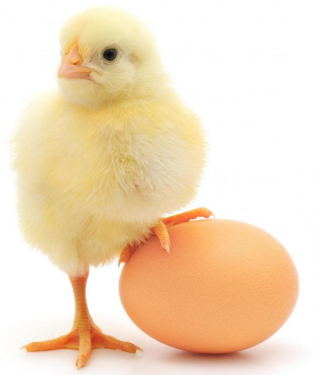

No! $(n^{1/2+o(1)}, \varepsilon)$-hop set only requires balls up to $n^{1/2+o(1)}$ hops

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Iterative computation

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Compute balls with k priorities in H_i up to $n^{2/k}$ hops

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Omitted detail: weighted graphs, use rounding technique

Beyond Hop Sets

New Distributed Algorithm

Theorem ([Becker/Karrenbauer/K/Lenzen arXiv'16])

There is a deterministic algorithm for computing $(1+\varepsilon)$ approximate SSSP in $(\sqrt{n} + Diam) poly(\log n, \varepsilon)$ rounds.

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Shortest Transshipment Problem

Find the cheapest route for sending units of a single good from sources to sinks along the edges of a graph as specified by demands on nodes.

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SSSP: source has demand $-(n-1)$, other nodes have demand 1

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We approximate $\|\cdot\|_\infty$ by soft-max:

$$\mathsf{lse}_\beta(x) := \frac{1}{\beta} \ln\left(\sum_{i\in[d]} \left(e^{\beta x_i} + e^{-\beta x_i}\right)\right)$$

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Algorithm at a glance:

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- No straightforward way of obtaining per-node guarantee

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Main contributions:

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Open problems:

- PRAM: improve Cohen's $m^{1+o(1)}$ work with polylog depth?
- Deterministic decremental SSSP algorithm

Tight and Tighter