Distributed Laplacian Solving with Applications

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SIROCCO 2022

Joint work with Gramoz Goranci, Yang P. Liu, Richard Peng, Xiaorui Sun, Tijn de Vos, and Mingquan Ye

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 947702). Supported by the Austrian Science Fund (FWF): P 32863-N

Laplacian Paradigm

- Laplacian systems
- Spectral sparsifiers
- Electrical flow
- Effective resistance
- Expander decompositions
- Continuous optimization
- Interior-point methods
- Gradient descent
- Preconditioning
- ...

Laplacian Paradigm and Distributed Computing

Observation

Laplacian paradigm often yields inherently parallelizable algorithms

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Basic operation:

- Vector v: each node represents a coordinate
- Matrix ?: each edge represents a non-zero entry
- Matrix-vector multiplication ?v: one round

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State of the art for (approximate) single-source shortest path, maximum flow, minimum-cost flow:
[Ghaffari, Karrenbauer, Kuhn, Lenzen, Patt-Shamir '15] [Becker, F, Karrenbauer, Lenzen '17] [Zuzic '21] [Anagnostides, Themis Gouleakis, Christoph Lenzen '21] [Zuzic, Goranci, Ye, Haeupler, Sun '22] [Rozhon, Grunau, Haeupler, Zuzic, Li '22]

Laplacian Systems

Goal

Solve linear system Jv ; ` such that J is a **Laplacian matrix**.

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Definition

The **Laplacian matrix** J&ʕ' of graph ʕ ; &ɔ*ᵇ*‾' is defined by

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Laplacian Systems

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Solve linear system $Lx = b$ such that L is a **Laplacian matrix**.

Definition

The **Laplacian matrix** $L(G)$ of graph $G = (V, E, w)$ is defined by

$$L(G)_{u,v} = \begin{cases} \sum_{(u,v)\in E} w_{u,v} & \text{if } u = v, \\ -w_{u,v} & \text{otherwise.} \end{cases}$$

High-precision solver: Approximation of solution x^* with x s.t.

$$\|x - x^*\|_{L(G)} \leq \epsilon \|b\|_{L(G)},$$

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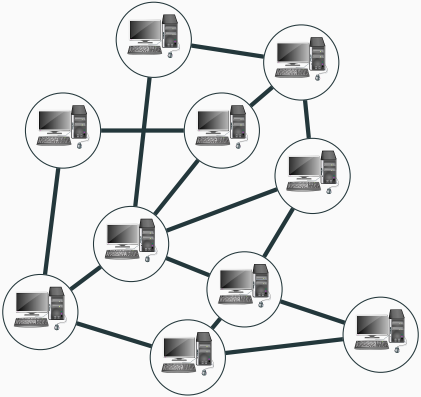
Prior work:

- $\tilde{O}(m)$ sequential running time [Spielman, Teng '04]
- $\tilde{O}(m)$ work, polylogarithmic depth [Peng, Spielman '14]

CONGEST Model



- Edges correspond to non-zero entries of matrix
- Each node holds one row/column of matrix
- Communication over edges in synchronous rounds
- Bandwidth ب&jmeم' per edge

Our Results for the CONGEST Model

Theorem ([F, Goranci, Liu, Peng, Sun, Ye])

Gl rfc AMLECQR kmbcj egtcl _ ucgefrcb _lb slbgpcarcb ep_nf ؟ _lb _ tcarmp ` ml م tcprgacq* uc a_l gl ب&مٴ&/' &₴م̄ ) ، '' pmslbq pcrspl _ tcarmp v qsaf rf_r ٌv € v₱ٌJ&؟' ╛ ٌ`ٌJ&؟',

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Implications

بہ ل1-5)ن&/'&م/-0 , /-2) ، ' -round algorithms in CONGEST model for the following problems:

- Maximum flow [Mądry ’16]
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First ن&م'-round algorithms for sparse, low-diameter graphs

Approximate Schur Complement

Definition (Schur complement)

For an م Õ م symmetric matrix K and a subset of *rcpkgl_jq* ح ◂ ۲م[, let ج ; ۲م[̂ ح, Permute the rows/columns of K to write

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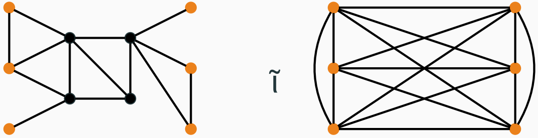
Graphical interpretation:



Input graph Schur complement

Approximate Schur Complement

Definition (Schur complement)

For an $n \times n$ symmetric matrix L and a subset of *terminals* $C \subseteq [n]$, let $F = [n] \setminus C$. Permute the rows/columns of L to write

$$L = \begin{pmatrix} L_{F,F} & L_{F,C} \\ L_{C,F} & L_{C,C} \end{pmatrix},$$

Then the *Schur complement* of L onto C is defined as $SC(L, C) := L_{C,C} - L_{C,F} L_{F,F}^{-1} L_{F,C}$.

Result of block Gaussian elimination

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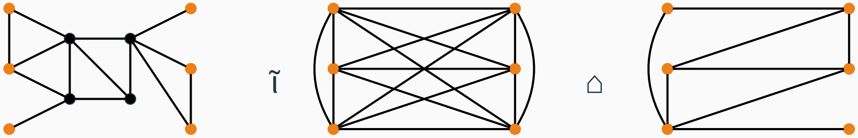

Input graph Schur complement Sparsification

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Communication of edges “along” sparsifier edges my lead to too much congestion

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Vertex sparsifiers as **minors** of the communication graph

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Key contribution: Parallel variant of [Li Schild ’18]

Technical Details

- From [Kyng, Lee, Peng, Sachdeva, Spielman ’16]: Repeated elimination of **almost independent sets** yields vertex sparsifier “chain” with recursion depth ط ; ب&jme م'

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  - Identify "steady" edges that can be sampled independently
  - Requires recursive solution of linear system: edge reduction via ultra-sparsifiers

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  - Identify “steady” edges that can be sampled independently
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  - Distortion of minor property in recursive calls

What Next?

Implications

بـ لـ1-5)ن&/'&م/-0 ، /-2) ،' -round algorithms in CONGEST model for the following problems:

- Maximum flow [Mądry '16]
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Sublinear #rounds in dense graphs?

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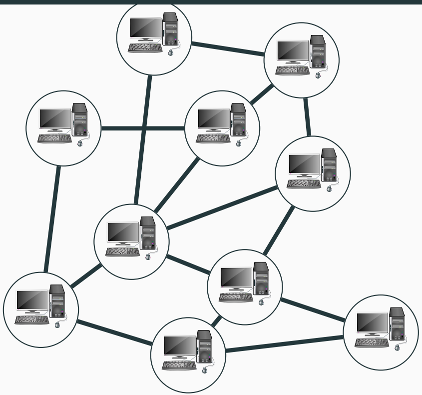
Sublinear #rounds in dense graphs?

Easier Question

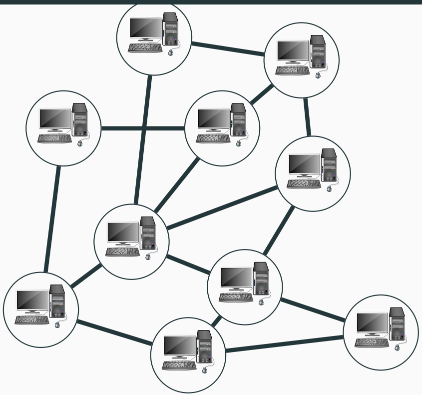
Sublinear #rounds on the Broadcast Congested Clique?

Broadcast Congested Clique



- Nodes can communicate with all other nodes [Lotker et al. ’05]
- Broadcast *rfc q_kc* message to all nodes [Drucker, Kuhn, Oshman ’12]

Broadcast Congested Clique



- Nodes can communicate with all other nodes [Lotker et al. '05]
- Broadcast *rfc q_kc* message to all nodes [Drucker, Kuhn, Oshman '12]
- For many problems: only "trivialization" of CONGEST model upper bounds with ， ; / is known

Our Results for the BCC

Theorem ([F, de Vos ’22])

Ml rfc @pm_ba_qr Amlecqrcb Ajgosc rfc kglgksk amqr jmu npm`jck*
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Other Results:

- On the Broadcast Congested Clique, a spectral sparsifier of quality / ¯ and size ب&م- 0' can be computed in ب&/- 0' rounds
- On the Broadcast Congested Clique, a Laplacian system can be solved up to high accuracy in ب&jme0&/- '' rounds
- On the Broadcast Congested Clique, certain Linear Programs can be solved in ب&ⱬ¯م' rounds.

Linear Programming

Minimize $a^R v$ subject to $?v \; ; \; `$

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Implementation of [Lee, Sidford ’14]:

- Interior point method with ب&ƻ$\overline{\text{rank}}$' iterations
- One linear system solve per iteration

Main Idea and Challenges

Linear Programming

Minimize $a^R v$ subject to ?v ; `

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Minimum cost flow:

- Rank = #nodes
- Linear system has Laplacian matrix

Key Contribution

Iterative computation of spectral sparsifier [Koutis, Xu '16]:

- Compute a spanner
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- Compute a spanner
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Problem

On Broadcast Congested Clique, nodes cannot easily coordinate with neighbors on sampling incident edges

Solution:

- Compute spanner on "probabilistic" graph
- Sample individual edges ad-hoc when needed
- Modification of spanner algorithm of [Baswana, Sen '07]

Optimization vs. Data Structures

[Lee, Sidford '14]:
#iterations: $\tilde{O}(\sqrt{n})$
Time per iteration: $\tilde{O}(m)$

[Chen et al. '22]:
#iterations: $m^{1+o(1)}$
Time per iteration: $m^{o(1)}$

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Is $\tilde{\Theta}(\sqrt{n})$ the right iteration count for min-cost flow LP?

Open Problem

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Is [illegible] the right round complexity for min-cost flow in the BCC?

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- Lower bounds in BCC at least not hopeless
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- Better upper bound already interesting for single-source reachability

Conclusion

Almost optimal Laplacian solvers

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Broadcast Congested Clique is an interesting “burning glass”