A Deterministic Almost-Tight Distributed Algorithm for Approximating Single-Source Shortest Paths

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Introduction

The problem:

- Single-source shortest paths
- Undirected graphs
- Positive edge weights $\in \{1, \ldots, poly(n)\}$
- Goal: $(1+\epsilon)$- or $(1+o(1))$-approximation ($\epsilon = 1/polylogn$)

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Distributed setting:

- Network modeled as undirected graph
- Processors can communicate with neighbors
- **CONGEST** model: synchronous rounds, message size $O(\log n)$
- Running time = number of rounds
- Goal: every node knows distance to source

Overview

Upper bounds:

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   Derandomization of "hitting paths" argument at cost of approximation
2. Compute hop set and approximate SSSP on overlay network
   Deterministic hop set using greedy hitting set heuristic

Summary of Results

Theorem (CONGEST)

There is a deterministic distributed algorithm that, on any weighted undirected network, computes $(1+o(1))$*-approximate shortest paths between a given source node s and every other node in* $O(n^{1/2+o(1)} + D^{1+o(1)})$ *rounds.*

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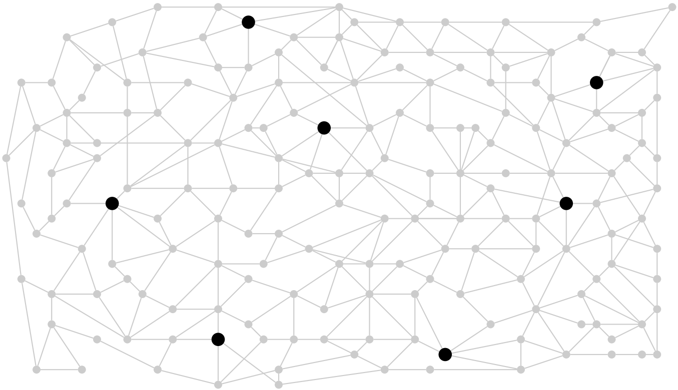
Theorem (Streaming)

There is a deterministic streaming algorithm that, given any weighted undirected graph, computes $(1 + o(1))$-approximate shortest paths between a given source node s and every other node in $O(n^{o(1)} \log W)$ passes with $O(n^{1+o(1)} \log W)$ space.

Computing Overlay Network

Overlay Network

Overlay Network



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   $\Rightarrow$ Every shortest path with $\geq \sqrt{n}$ edges contains center whp

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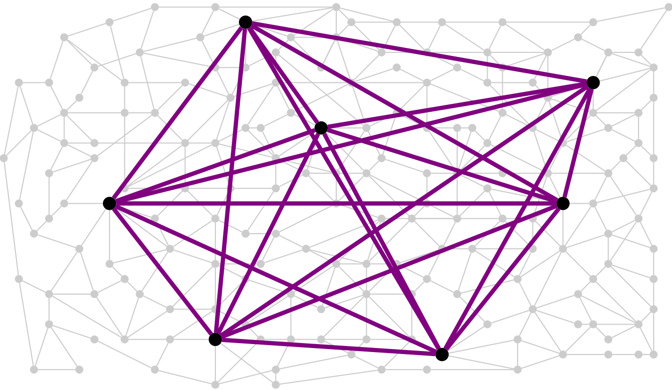

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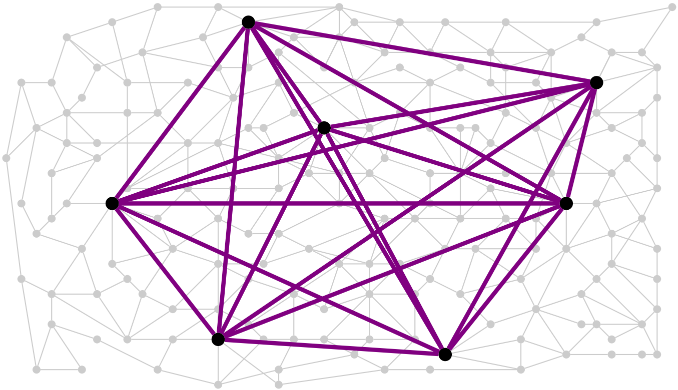

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3. Sufficient to solve SSSP on overlay network using hop set

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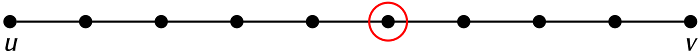
Property from randomization

$O(\sqrt{n} \log n)$ centers that hit every shortest path with $\geq \sqrt{n}$ edges



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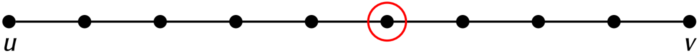
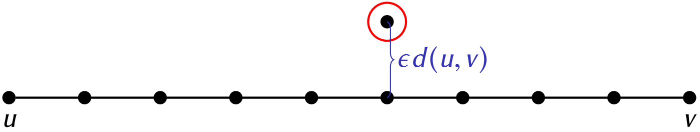


Deterministic relaxation

$O(\sqrt{n}\epsilon^{-1} \log n)$ centers that **almost** hit every path with $\geq \sqrt{n}$ edges



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Crucial: "weight = #edges" in unweighted graphs

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Well-known weight rounding [Bernstein '09/13, Madry '10, ...]

G_i: round up edge weights to next multiple of $\epsilon 2^i / \sqrt{n}$ ($\forall i = 1$ to $\log(nW)$)
$(1+\epsilon)$-approximation of shortest paths with $\sqrt{n}$ edges and weight $2^i \ldots 2^{i+1}$
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⇒ Determine centers by computing ruling set for all type classes

Computing Hop Set on Overlay Network

Hop Sets

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An (h, ϵ)-hop set is a set of weighted edges F such that, for all pairs of nodes u and v, in the 'shortcut graph' $G \cup F$ there is a path from u to v with **at most h edges** of weight at most $(1+\epsilon)dist(u, v)$.

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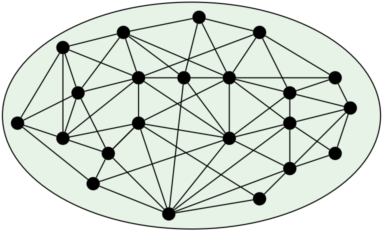

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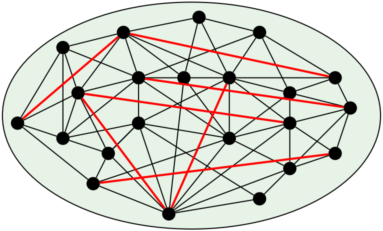

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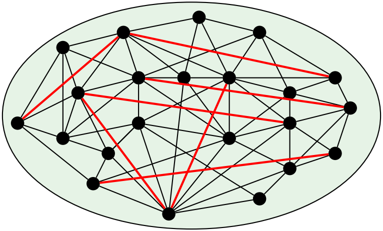

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A: $(\log^{O(1)} n, \epsilon)$-hop set of size $n^{1+o(1)}$ [Cohen '94]
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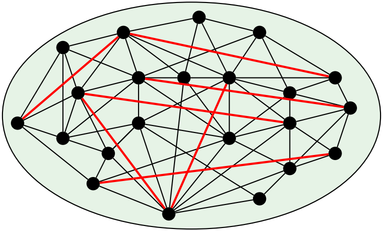

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Our contribution: Fast computation of **B** on overlay network

Hop Set Based on Clusters [Thorup/Zwick '01]

$V = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_k = \emptyset$ where node of A_i goes to A_{i+1} with probability $1/n^{1/k}$

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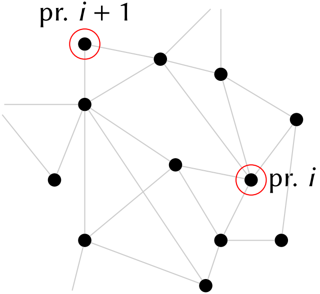

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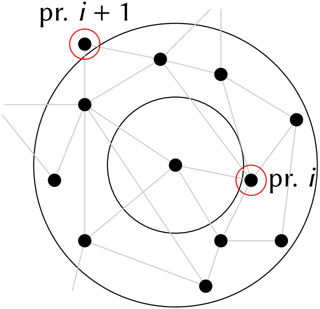

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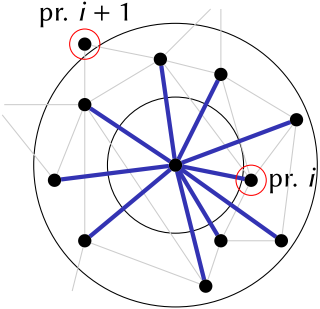

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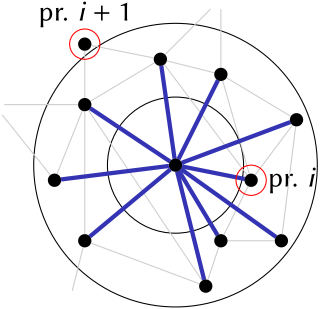

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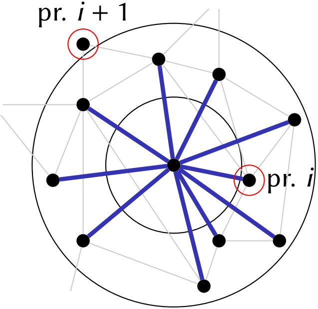

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- With $k = \sqrt{\log n}/\sqrt{\log 4/\epsilon}$: $(n^{o(1)}, \epsilon)$-hop set of size $n^{1+o(1)}$

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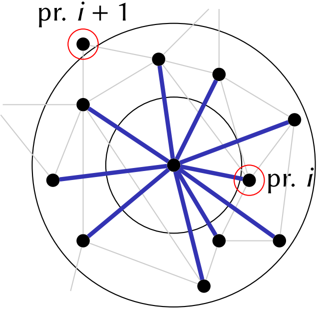

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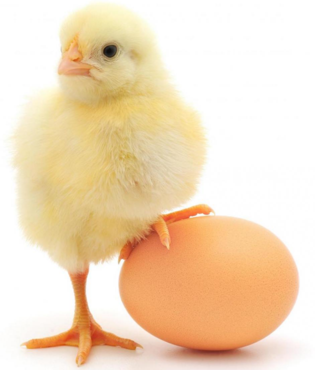
- With $k = \sqrt{\log n}/\sqrt{\log 4/\epsilon}$: $(n^{o(1)}, \epsilon)$-hop set of size $n^{1+o(1)}$
- **Derandomization:** choose A_{i+1} from A_i by greedy hitting set heuristic *(Sequential, but affordable in overlay network)*

Chicken-Egg Problem?

1. Goal: Faster SSSP via hop set
2. Compute hop set by computing clusters
3. Computing clusters at least as hard as SSSP

⇒ Back at problem we wanted to solve initially?



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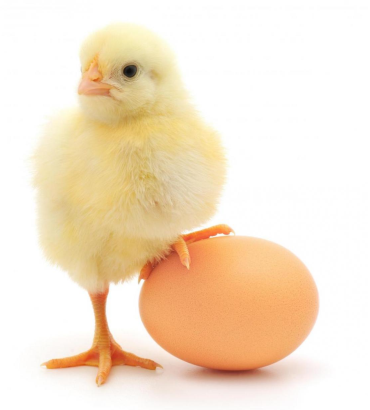

No! Iterative computation starting with

- SSSP up to small #hops is cheap in overlay network
- Clusters up to small #hops provide sufficient shortcutting to make progress in each iteration

Computing $(n^{o(1)}, \epsilon)$-hop set

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Compute clusters with k priorities in H_i up to $n^{2/k}$ hops

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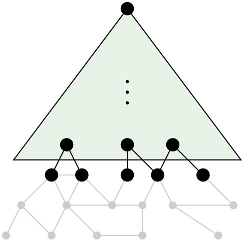
Omitted detail: weighted graphs, use rounding technique

Computing Hop Set on Overlay Network

Shortest paths from source *s* **up to distance** d**:**



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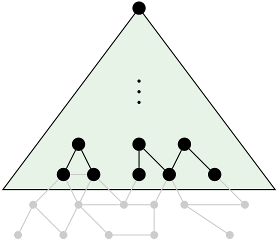

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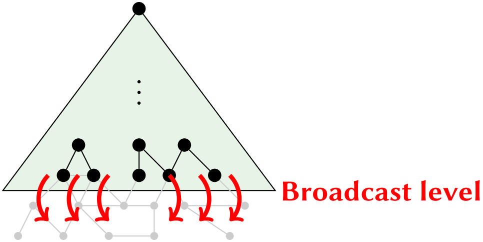

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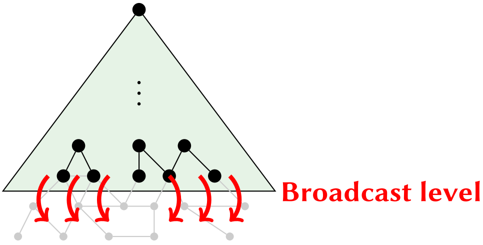

d iterations, each $O(Diam + N_\ell)$ rounds where N_ℓ = #nodes at level ℓ
Running time: $O(d \cdot Diam + \sum_{l \leq d} N_\ell) = O(d \cdot Diam + N)$

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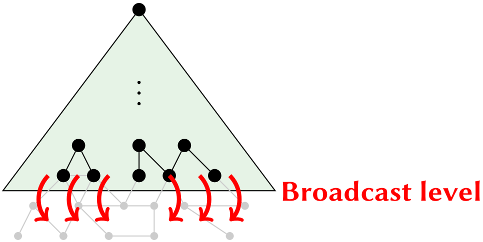

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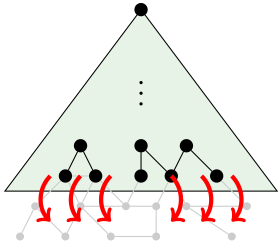

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$\Rightarrow$ Hop Set and approximate SSSP: $O(n^{1/2+o(1)} + Diam^{1+o(1)})$ $(N \approx \sqrt{n})$

Conclusion

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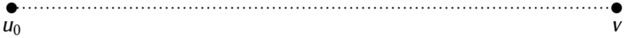
Open problems:

- $n^{o(1)} \rightarrow \log^{O(1)} n$
  Better hop set?
- Improve dependence on ϵ
- $O(n)$ rounds optimal for exact SSSP?

Example: $(n^{1/2+o(1)}, \epsilon)$-hop set

Case 1: $dist(u_0, v) \leq n^{1/2+1/k}/\epsilon$



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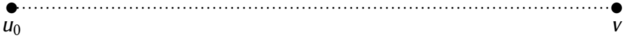

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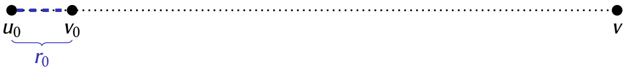

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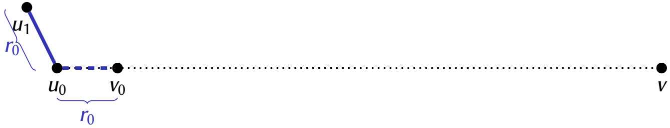

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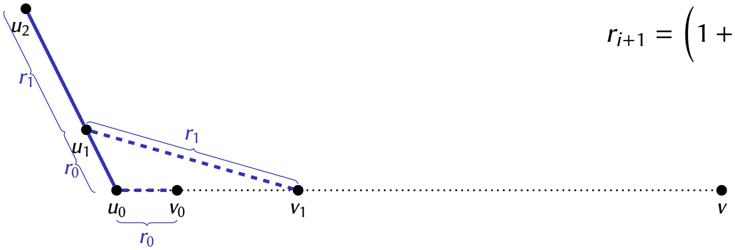

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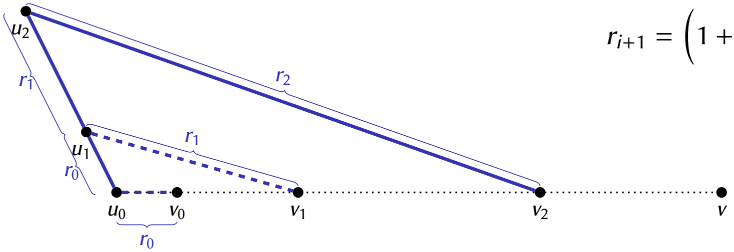

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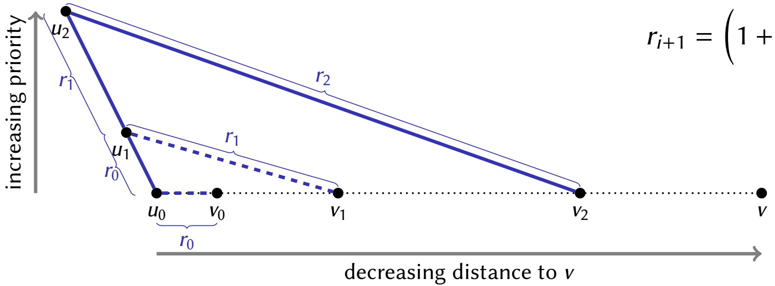

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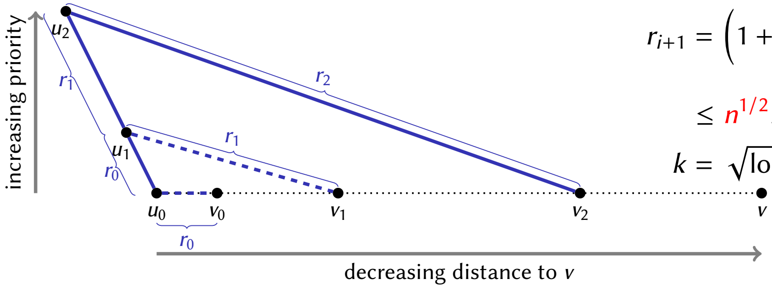

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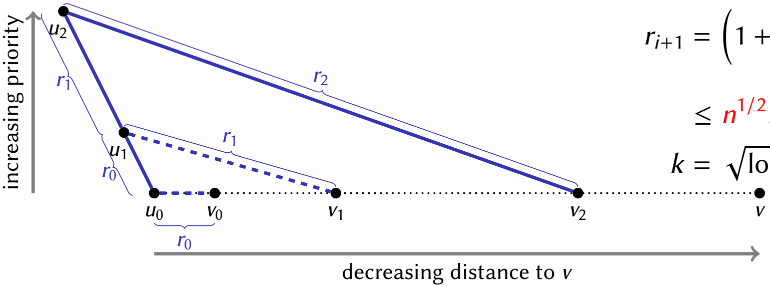

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$$\#\textit{Edges} \le \frac{k \cdot dist(u, v)}{n^{1/2}} \le \frac{k \cdot n}{n^{1/2}} = kn^{1/2}$$