Fully dynamic all-pairs shortest paths with worst-case update-time revisited

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Joint work with Ittai Abraham and Shiri Chechik

Research Overview

	shortest paths	cuts	verification	lower bounds
dynamic				
distributed				
sequential				

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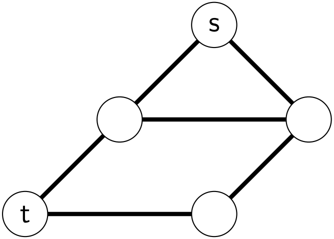
Simple example – Distance from s to v



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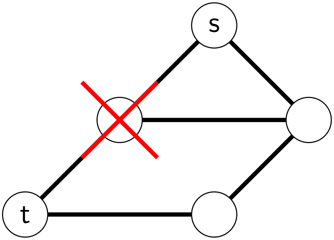

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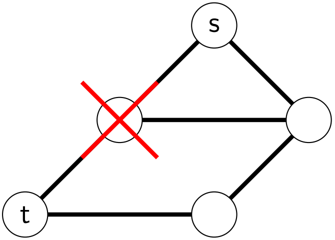

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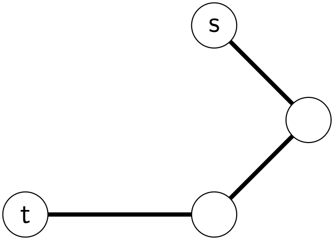

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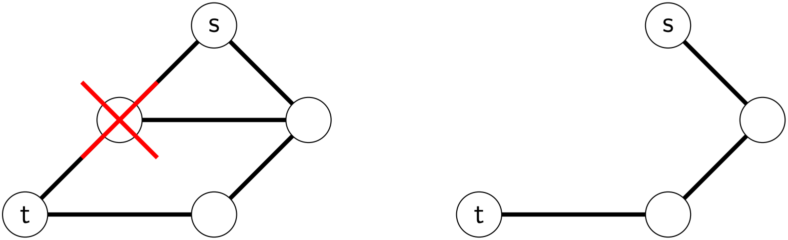


dist(s, t) $= 2$ dist(s, t) $= 3$

Dynamic shortest paths data structure:

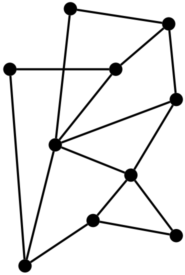
- initialize(G)
- insert(v)
- delete(v)
- dist(s, t)
- path(s, t)

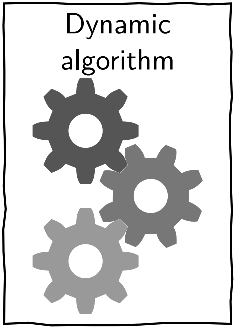
$\left.\begin{array}{l}\text{insert}(v)\\ \text{delete}(v)\end{array}\right\}$ update

$\left.\begin{array}{l}\text{dist}(s,\ t)\\ \text{path}(s,\ t)\end{array}\right\}$ query

Dynamic model

G undergoing updates:





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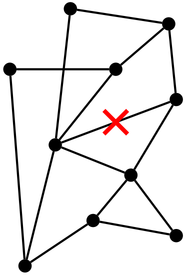

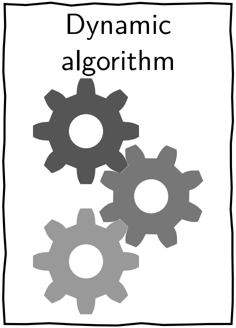

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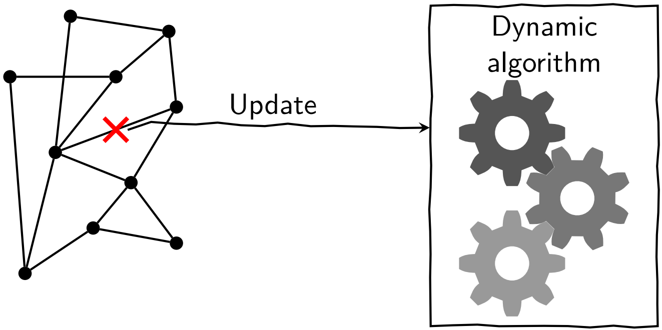

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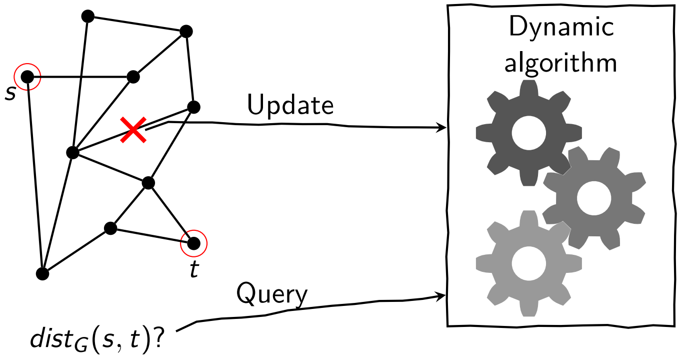

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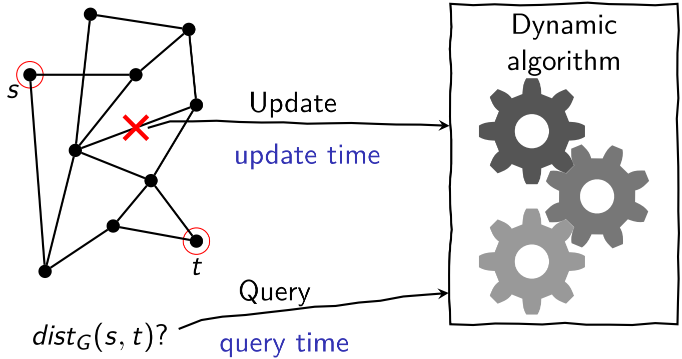

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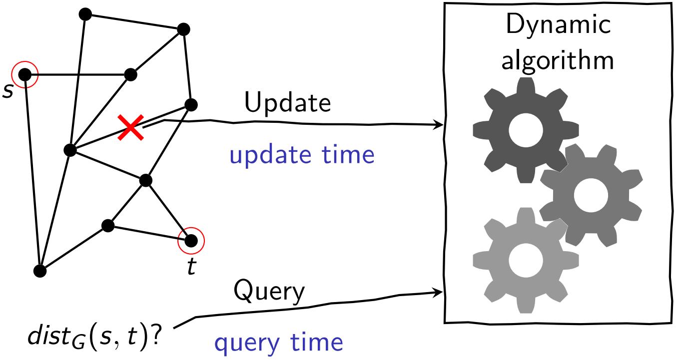

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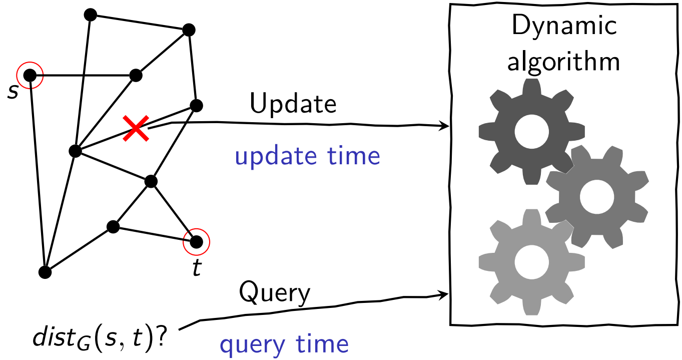

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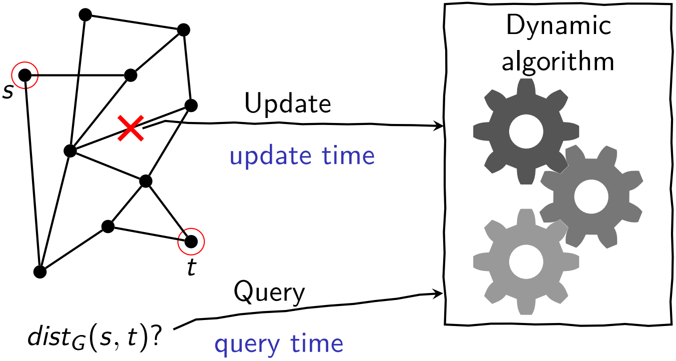

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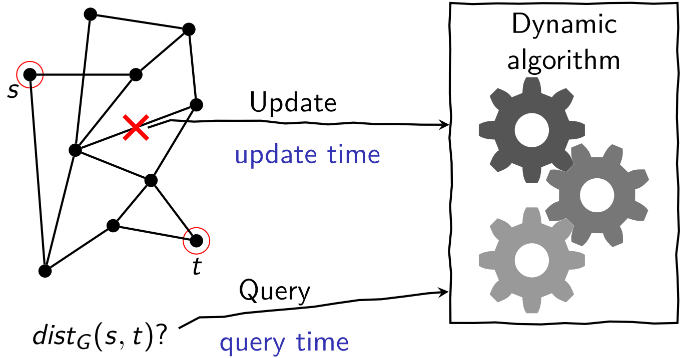

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- Worst-case: After each update, spend time $\leq T(n, m)$
- Amortized: For a sequence of k updates, spend time $\leq kT(n, m)$

Overview

	amortized	**worst-case**
Connectivity	$\log^{\mathcal{O}(1)} n$ [Henzinger/King '95]	$\log^{\mathcal{O}(1)} n$ [Kapron/King/Mountjoy '13]

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$(1+\epsilon)$-cut sparsifier	$\log^{\mathcal{O}(1)} n$ [Abr./Durfee/Koutis/K/Peng '16]	$\log^{\mathcal{O}(1)} n$ [Abr./Durfee/Koutis/K/Peng '16]

Question:
Can worst-case bounds match amortized bounds?

Prior work on dynamic APSP

approx.	update time	type of graphs	reference
exact	$\tilde{\mathcal{O}}(mn)$	weighted directed	[Dijkstra]
exact	$\tilde{\mathcal{O}}(n^{2.5}\sqrt{W})$	weighted directed	[King '99]
$1+\epsilon$	$\tilde{\mathcal{O}}(n^2 \log W)$	weighted directed	[King '99]
$2+\epsilon$	$\tilde{\mathcal{O}}(n^2)$	weighted directed	[King '99]
exact	$\tilde{\mathcal{O}}(n^{2.5}\sqrt{W})$	weighted directed	[Demetrescu/Italiano '01]
exact	$\tilde{\mathcal{O}}(n^2)$	weighted directed	[Demetrescu/Italiano '03]
exact	$\tilde{\mathcal{O}}(n^{2.75})$ (*)	weighted directed	[Thorup '05]
$2+\epsilon$	$\tilde{\mathcal{O}}(m \log W)$	weighted undirected	[Bernstein '09]
$2^{\mathcal{O}(k)}$	$\tilde{\mathcal{O}}(\sqrt{m} n^{1/k})$	unweighted undirected	[Abr./Chechik/Talwar '14]

(*) worst case

$\tilde{\mathcal{O}}$: ignores $\log n$-factors
n: number of nodes
m: number of edges
W: largest edge weight

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Theorem (for this talk)

There is an algorithm for maintaining a distance matrix under insertions and deletions of nodes in unweighted undirected graphs with a worst-case update time of $\tilde{\mathcal{O}}(n^{2.75})$.

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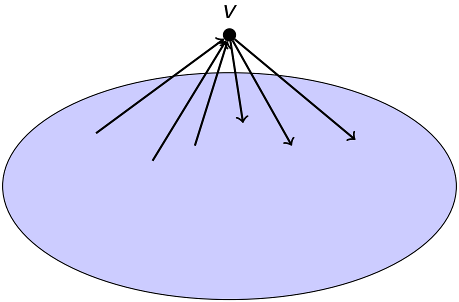
More sophisticated use of our technique:

- $\tilde{\mathcal{O}}(n^{2.67})$ in weighted directed graphs (randomized)
- Improves $\tilde{\mathcal{O}}(n^{2.75})$ of [Thorup '05]
- (Hopefully) simpler than [Thorup '05]
  (which is a deamortization of [Demetrescu/Italiano '03])

Insertions are easy

Inserting a node v:



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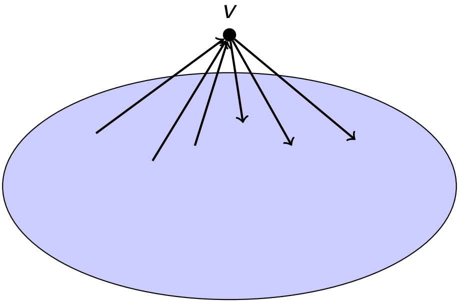

Floyd-Warshall algorithm

- For every node s: $dist'(s,v) = \min_{(u,v)}(dist(s,u) + w(u,v))$
- For every node t: $dist'(v,t) = \min_{(v,u)}(w(v,u) + dist(u,t))$
- For every pair s,t: $dist'(s,t) = \min(dist(s,t), dist'(s,v) + dist'(v,t))$
- Time per insertion: $\mathcal{O}(n^2)$

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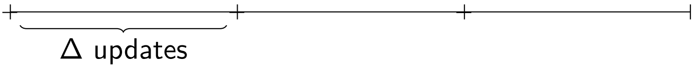

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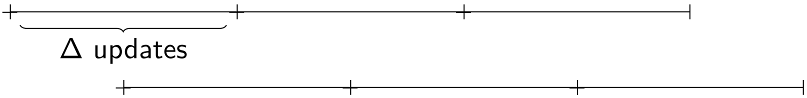

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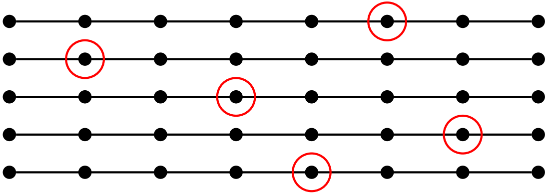

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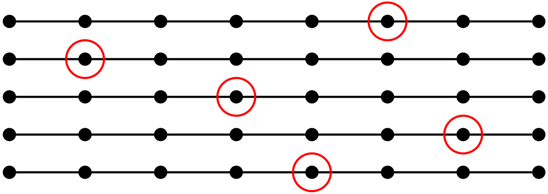

- Find hitting set C of size $\mathcal{O}(n \log n/h)$ in time $\mathcal{O}(n^2 h)$ (greedy)

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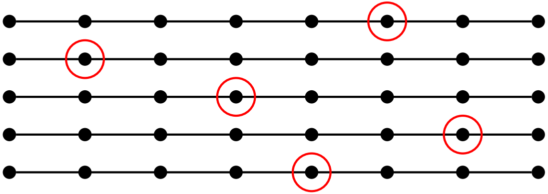

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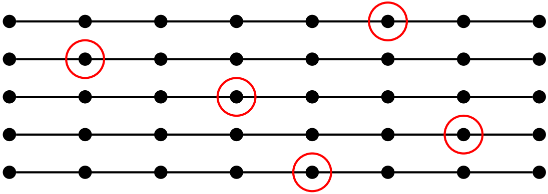

- Find hitting set C of size $\mathcal{O}(n \log n/h)$ in time $\mathcal{O}(n^2 h)$ (greedy)
- Compute shortest paths from nodes in C: $\mathcal{O}(n^3 \log n/h)$
- For all pairs s, t:
  $dist(s,t) = \min(dist^h(s,t), \min_{v \in C}(dist(s,v) + dist(v,t)))$

Summary

Known techniques allow the following restrictions:

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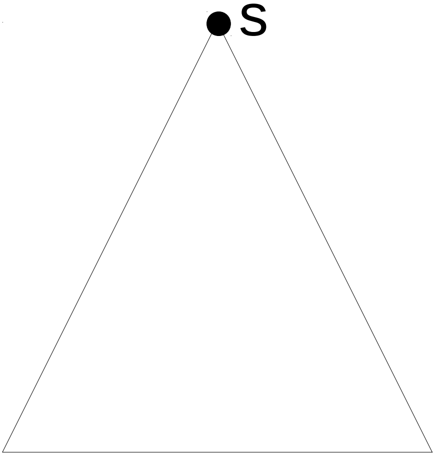
Known techniques allow the following restrictions:

1. Only necessary to maintain shortest h-hop paths up to length (for some parameter h)
2. To obtain a fully dynamic algorithm it is sufficient to design a deletions-only algorithm that
   - can handle up to Δ deletions of nodes with worst-case guarantees
   - after **preprocessing** the graph

   Restart deletions-only algorithm each Δ updates

Repairing a shortest path tree



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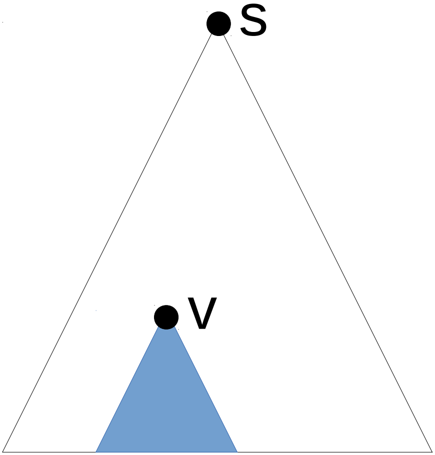

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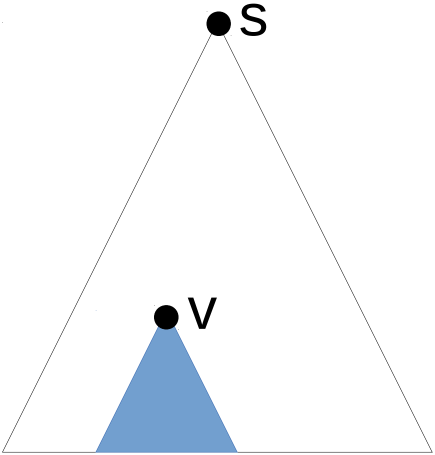

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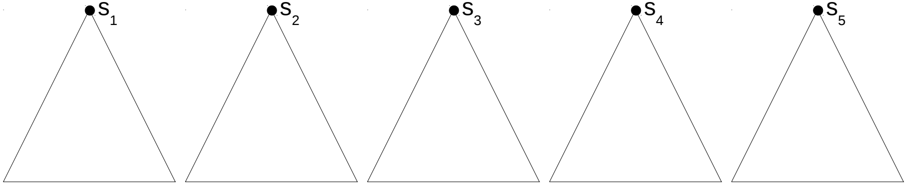
- Run Dijkstra's algorithm to reattach these nodes to the tree
- Charge time $\mathcal{O}(deg(u)) \leq \mathcal{O}(n)$ to every node u in subtree of v

Multiple shortest path trees

Goal: shortest paths from a set of source nodes S



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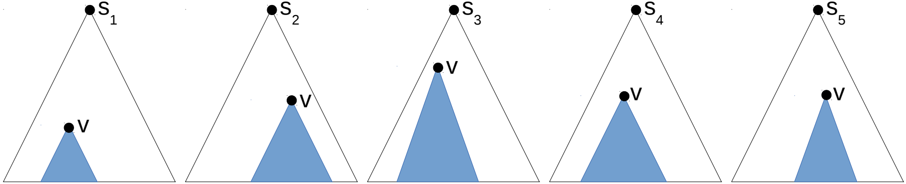

Deletion of v

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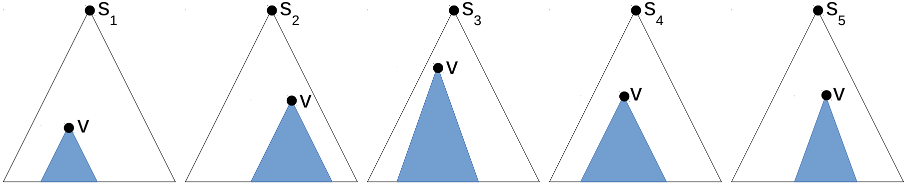

Deletion of v

Total work: (number of nodes in subtrees of v) $\times n$

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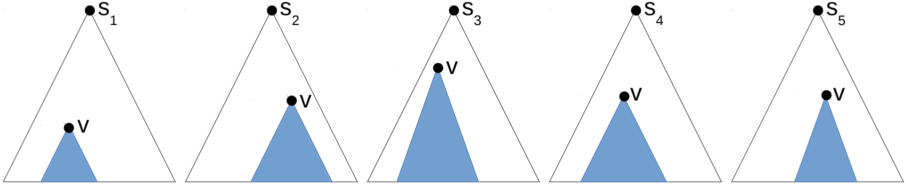

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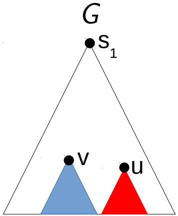
Goal: limit sizes of subtrees of each node

Preprocessing

Construct shortest path tree up to depth h for all sources one by one:



Count size of subtrees for every node

Rule: If number of nodes in subtrees of v exceeds λ:

- v is added to set of **heavy** nodes H
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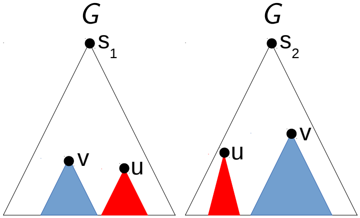

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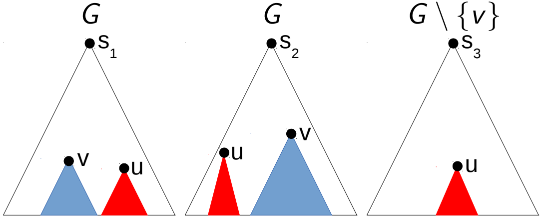

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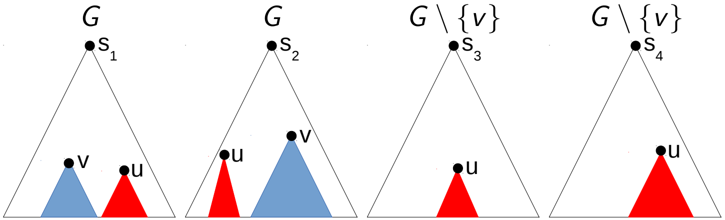

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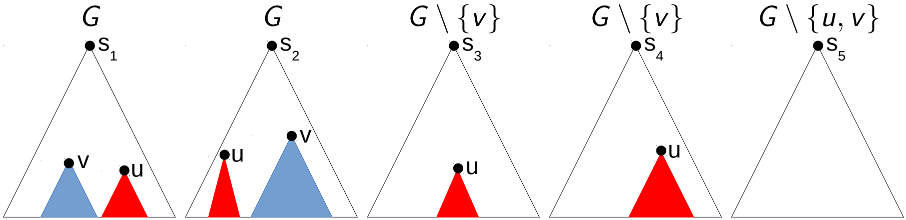

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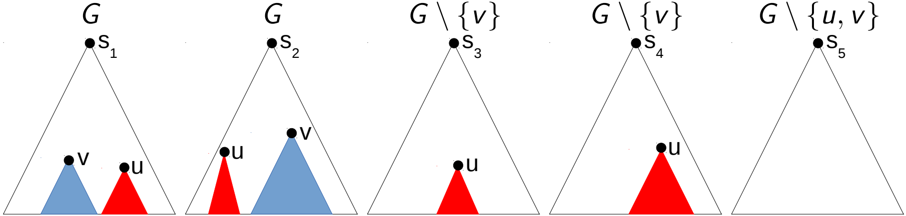

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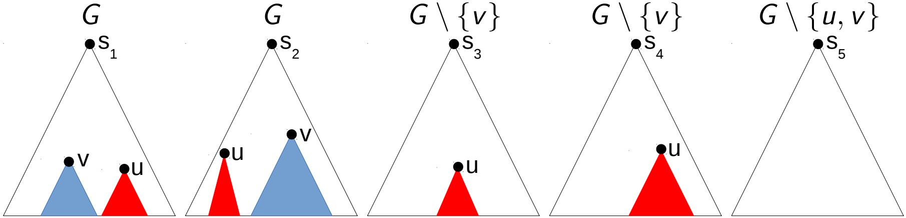
Observations:

- All shortest paths not using heavy nodes included in trees
- Number of heavy nodes: $|H| \leq \mathcal{O}(\frac{|S|nh}{\lambda}) \leq \mathcal{O}(\frac{n^2 h}{\lambda})$
- Preprocessing time: $\mathcal{O}(|S|n^2) \leq \mathcal{O}(n^3)$

Computing distances after Δ deletions



1. For all deleted nodes: Reattach children to tree using Dijkstra
   Running time: $\mathcal{O}(\Delta\lambda n)$ per deletion
   - Subtree size at most λ per node
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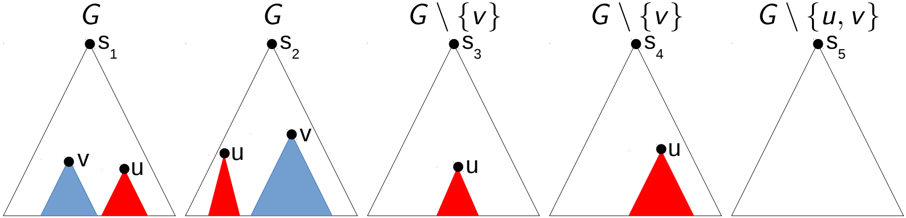

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   Correct for all shortest paths not containing heavy nodes
2. Special treatment of heavy nodes: shortest paths via heavy nodes

   $$\text{Compute } \min_{v \in H}(dist(s, v) + dist(v, t)) \text{ for all } s \text{ and } t$$

   Time per deletion: $\mathcal{O}(|H|n^2) = \mathcal{O}(\frac{n^4 h}{\lambda})$

Running time wrapped up

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- $\tilde{\mathcal{O}}(n^2 h + \frac{n^3}{h})$ Shortest paths of length more than h

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$$\Delta = n^{0.25},\ \lambda = n^{1.5},\ h = n^{0.25}$$
$$\Rightarrow \tilde{\mathcal{O}}(n^{2.75})$$

Improvements

Directed graphs:
Two types of shortest path trees: incoming and outgoing

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Increased efficiency:
Multiple instances of algorithm to cover all hop ranges (+randomization)
Load balancing trick

Barriers

Combinatorial approach [Thorup '05, Abraham/Chechik/Krinninger '17]

The best we can hope for:

- Preprocessing: $\mathcal{O}(n^3)$
- Spread preprocessing over Δ updates: $\mathcal{O}(n^3/k)$
- Deal with $\leq \Delta$ insertions after each update: $\mathcal{O}(n^2 k)$

$\Rightarrow \mathcal{O}(n^{2.5})$

Algebraic approach [Sankowski '04/'05]

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Transitive closure:

- Count number of paths from s to t for all pairs
- Reachable iff $\#\text{paths} > 0$
- Perform operations for counting modulo random prime
- Update time $\mathcal{O}(n^2)$
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All-pairs shortest paths (distances):

- For every $1 \leq \ell \leq h$, count #paths of length exactly ℓ
- Additional trick: fast convolution
- Update time: $\tilde{\mathcal{O}}(n^2 h)$.
- Standard trick for hitting long paths: $h = \sqrt{n}$

$\Rightarrow \mathcal{O}(n^{2.5})$

Open problems

How about the following worst-case bounds?

- Fully dynamic APSP: Meet $n^{2.5}$ barrier
- Fully dynamic APSP: $(1+\epsilon)$-approximation in $\tilde{\mathcal{O}}(n^2/\epsilon)$ time?
- Fully dynamic transitive closure: deterministic $\mathcal{O}(n^2)$ algorithm?