# Orthogonal Vectors Problem

Def OVP

<u>Given</u>: Set $A, B \in \{0,1\}^d$, $n = |A| = |B|$

<u>Task</u>: Decide if there $a \in A$ $b \in B$ s.t. $a \perp b$

$\langle a, b \rangle = 0$

$\forall\, 1 \le i \le d$: $a[i] = 0$ and $b[i] = 0$ or $a[i] = 0$ and $b[i] = 1$ or $a[1] = 1$ and $b[i] = 0$

Algorithms
- $O(n^2 d)$ trivial
- Fastest known $O(n^{2 - 1/O(\log c)})$ if $d = c \log n$

OPEN Is there an $O(n^{2-\varepsilon}\,\mathrm{poly}(d))$-time alg for some constant $\varepsilon > 0$

<u>Theorem</u>: If there was an $O(mn^{1-\varepsilon})$-Time alg for diameter, then there would be an $O(n^{2-\delta}\,\mathrm{poly}(d))$-time alg for OV.

new alg for OV

OV instance $(A,B)$ $\xrightarrow[G_{A,B}]{\text{reduction}}$ $\boxed{\text{Diam. alg.}}$

$(A,B)$ has orth pair iff $G_{A,B}$ has diameter 3

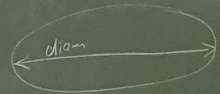
$G_{A,B}$ has size $O(n \cdot d)$

(& can be constructed in time $O(n \cdot d)$)

<u>Def</u>: Diameter Problem

Given (Unweighted) Graph $G = (V, E)$ $n = |V|$ $m = |E|$

Task: Compute diameter of $G$, $\mathrm{diam}(G) = \max_{u,v \in V} \mathrm{dist}_G(u,v)$




Simple alg:
- BFS for every node $u$ (BFS gives $\mathrm{dist}(u, \cdot)$) $O(m)$
- Determine max over all pairs $O(n^2)$

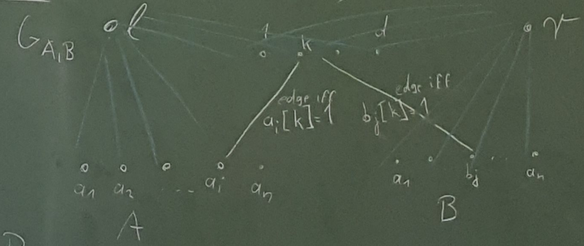
$O(mn)$

OPEN $O(mn^{1-\varepsilon})$-alg for diameter?

<u>Proof</u>: [Idea $\mathrm{diam} = 2 \Leftrightarrow$ no orth pair]

if $\exists$ orth pair $\Rightarrow$ diam = 3

if $\nexists$ orth pair $\Rightarrow$ diam = 2




Preprocessing step: make sure every $a_i \in A$ has at least one $k$ with $a_i[k] = 1$ (if not $a_i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ orth to every $b_j$)

Observe: $d(u,v) \le 3$ $\forall u, v \in V$

More precisely $d(u,v) \le 3$ if $u \in A$, $v \in B$

$d(u,v) \le 2$ o.w.

If no orth pair: $\forall a \in A$, $b \in B$ there is $k$ s.t. $a[k] = 1 = b[k]$

$\Rightarrow \exists$ path $a - k - b$ length 2

$\Rightarrow \mathrm{dist}(a,b) \le 2$

With Observation

$\Rightarrow \mathrm{dist}(u,v) \le 2$ $\forall u, v$

$\Rightarrow \mathrm{diam}(G) \le 2$

$\mathrm{diam}(G) = 2$ (bec $d(\ell, r) > 1$)

# Proof ct'd

$(*) = O(n^{2-\varepsilon}\,\mathrm{poly}(d))$

If there is orth. pair, say $a \in A$, $b \in B$

whenever $a[k]=1$, we have $b[k]=0$

We check that no path from $a$ to $b$ has length $<3$

Enumerate all possibilities (simple paths)

- $a - \ell - a' - k$ (for some $a' \in A$ and some $k$)
- $a - \ell - k - r$ (for some $k$)
- $a - k - b' - \dots$ then $b' \neq b$

None of those paths of length $<3$ above includes $b$

$\Rightarrow \mathrm{dist}(a,b) \geq 3 \overset{\text{with Obs}}{\Longrightarrow} \mathrm{diam}(G) \geq 3 \Rightarrow \mathrm{diam}(G) = 3$

Time Complexity:

$G_{A,B}$ has $N = 2n + d + 2 = O(n+d)$

$M \leq 2N + 2d + 2n \cdot d$

Running time of our alg.: $= O(n \cdot d)$

$O(M+N) + O(M N^{1-\varepsilon}) = O(n \cdot d \cdot (n+d)^{1-\varepsilon})$

$= (*)$