Node-Weighted Negative Triangle Detection

Triangle $\longrightarrow$ Node-Weighted Negative Triangle $\longrightarrow$ Edge-Weighted Negative Triangle

$O(n^{\omega})$ | $O(n^{\omega})$ [Czumaj Lingas '07] (actually minimum node-weighted triangle) | $O(n^3)$

Node-Weighted $\longrightarrow$ Edge-Weighted

Given node weight $w(i)$ for each node i

For each edge (i,j) set edge weight

$$w(i,j) := \frac{w(i) + w(j)}{2}$$

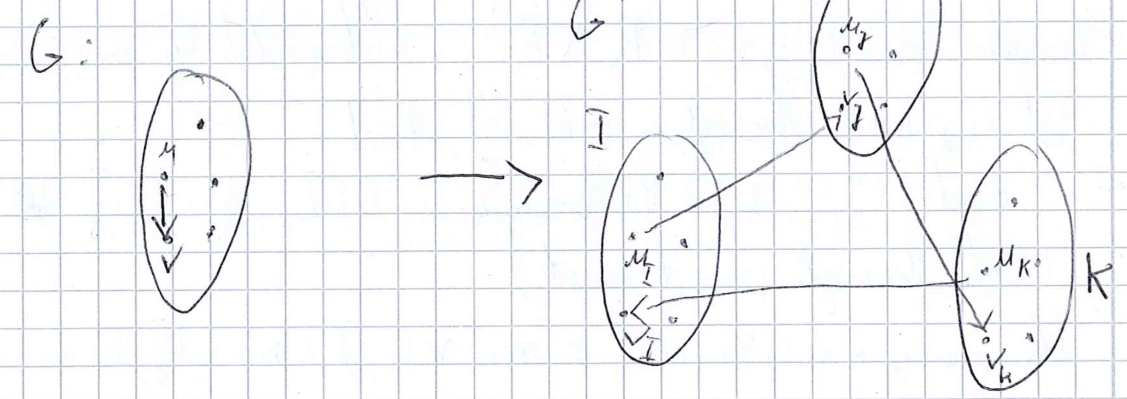
Observation: For every triangle (i,j,k):

$$w(i,j) + w(j,k) + w(k,i) = \frac{2w(i) + 2w(j) + 2w(k)}{2}$$

$$= w(i) + w(j) + w(k)$$

Algorithm for Node-Weighted Min Triangle

Assume w.l.o.g. that graph is tripartite:



triangle i, j, k $\iff$ triangle i_I, j_J, k_K

Assume w.l.o.g. that I, J, K are sorted by weight

Given tripartite $G=(V,E)$ with $V = I \cup J \cup K$ and node weights $w(v)$

Goal: Find minimum q s.t. $\exists\, i \in I, j \in J, k \in K$ s.t. $(i,j), (j,k), (k,i) \in E$ and $w(i) + w(j) + w(k) = 0$

Recursive algorithm ALG(G)

- global parameter p, a sufficiently large constant
- assume $n := |I| = |J| = |K| = p^l$ for some $l \in \mathbb{N}$ (o.w. add isolated dummy nodes to fill up)

0) If $n = O(1)$ $n \leq p^3$, then solve instance in constant time $O(1)$

1) Split $I = I_1 \cup \dots \cup I_p$, $J = J_1 \cup \dots \cup J_p$, $K = K_1 \cup \dots \cup K_p$ $O(n \log n)$
in sorted order $\max_{i \in I_x} w(i) \leq \min_{i \in I_{x+1}} w(i)$ etc.

2) Determine $R := \{(x,y,z) \in \{1,\dots,p\}^3 \mid G[I_x \cup J_y \cup K_z]$ contains a triangle $\}$ $O(p^3 n^\omega)$

3) Determine $R' := \{(x,y,z) \in R \mid$ there is no $(x',y',z') \in R$ with $x' < x, y' < y, z' < z\}$ (Pareto-optimal indices) $O(p^6)$

4) For each $(x,y,z) \in R'$: run $ALG(G[I_x \cup J_y \cup K_z])$ $\frac{n}{p}$ nodes
(and return min. among all recursive calls)

<u>Correctness</u>

We show that the algorithm does not "dismiss" any min-weight triangle.
For every triangle (i,j,k) we have $i \in I_x$, $j \in J_y$, $k \in K_z$ for some $(x,y,z) \in \{1,\dots,p\}^3$
($\Rightarrow$ (i,j,k) not dismissed after Step 2)

Consider some $(x,y,z) \in R \setminus R'$, i.e. dominated by some $(x',y',z') \in R'$
Let (i,j,k) be triangle in $G[I_x \cup J_y \cup K_z]$
and (i',j',k') be triangle in $G[I_{x'}, J_{y'}, K_{z'}]$
(both triangles must exist)

Then $w(i) + w(j) + w(k) \geq \min w(I_x) + \min w(J_y) + \min w(K_z)$

$$\rotatebox{90}{$\leq$} \qquad \rotatebox{90}{$\leq$} \qquad \rotatebox{90}{$\leq$}$$

$w(i') + w(j') + w(k') \leq \max w(I_{x'}) + \max w(J_{y'}) + \max w(K_{z'})$

$\Rightarrow$ (i,j,k) was not of minimum weight and may therefore be dismissed

Running Time Analysis:

~~A~~ Triangle Detection: $O(n^\omega)$ — assume $p^3 \le n$ otherwise $n = O(1)$

Recursive call: size $\frac{n}{p}$ — $\Rightarrow O(p^6) = O(n^2) = O(n^\omega)$

Question: How many recursive calls?
= How large is R'?

Lemma: $|R'| \le 4p^2$

Proof: ~~For~~ For every $r, s \in \{-p, \dots, p\}$ define

$$\Delta_{r,s} = \{(x,y,z) \in \{1,\dots,p\}^3 \mid x-y = r \text{ and } x-z = s\}$$
$$= \{(1, 1-r, 1-s), (2, 2-r, 2-s), \dots, (p, p-r, p-s)\} \cap \{1,\dots,p\}^3$$

Observation: $\{1,\dots,p\}^3 = \bigcup_{r,s \in \{-p,\dots,p\}} \Delta_{r,s}$

"$\supseteq$" ✓

"$\subseteq$" Let $(x,y,z) \in \{1,\dots,p\}^3$

Set $r = x - y$ $\quad s = x - z$ $\quad r, s \in \{-p,\dots,p\}$

$\Rightarrow (x,y,z) \in \Delta_{r,s}$

~~We define on an any injective mapping~~ $\cancel{f: R' \to \{-p,\dots,p\}^2}$
$\cancel{f(x,y,z) = (r,s)}$ ~~with~~

~~Each set $\Delta_{r,s}$ has a unique minimal element~~

In $\Delta_{r,s}$, each element is completely determined by its first coordinate

$\Rightarrow$ There is a total order on elements in $\Delta_{r,s}$ ~~for each~~
(according to first coordinate)

$\Rightarrow$ Each element of R' can be contained in at most one ~~ste~~ set $\Delta_{r,s}$ (for some $r, s \in \{-p,\dots,p\}$)

(Formally: injective mapping from R' to $\{-p,\dots,p\}^2$)

$\Rightarrow |R'| \le |\{-p,\dots,p\}^2| \le (2p)^2 \le 4p^2$

Recursion:

$$T(n) \leq 4p^2 \cdot T(n/p) + O(p^3 n^\omega)$$

$$T(n) \leq 4p^2 \cdot T(n/p) + \alpha p^3 n^\omega$$ for some constant α

Claim: $T(n) \leq 2\alpha p^3 n^\omega$

Proof: $T(n) \leq 4p^2 \cdot 2\alpha p^3 (n/p)^\omega + \alpha p^3 n^\omega$

$$= \alpha p^3 n^\omega (1 + 8p^{2-\omega})$$

If $\omega > 2$ $\leq 2\alpha p^3 n^\omega$ for sufficiently large constant p

$\Rightarrow T(n) \leq O(n^\omega)$

If $\omega = 2$, one can show $T(n) \leq 2\alpha p^3 n^{\omega+\varepsilon}$ for any $\varepsilon > 0$

$$\Rightarrow T(n) \leq O(n^\omega + n^{2+\varepsilon})$$