Fast Deterministic Fully Dynamic Distance Approximation

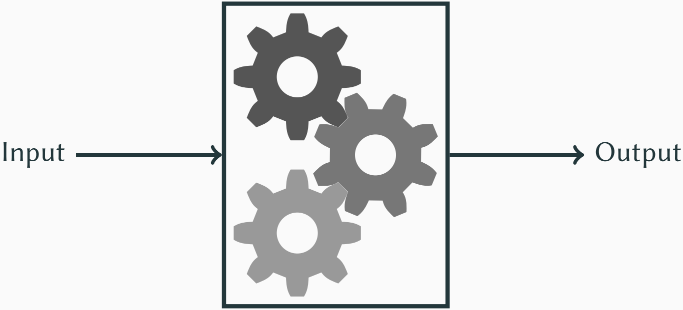
Sebastian Forster

Meeting on Algorithmic Challenges of Big Data 2022

University of Salzburg

Joint work with: Jan van den Brand   Yasamin Nazari

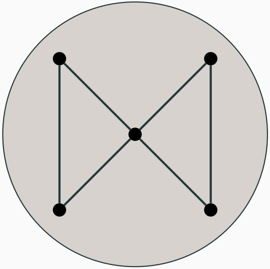
Static Approach



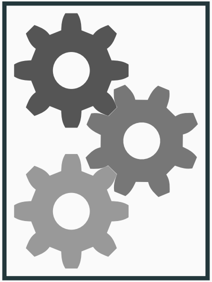
Dynamic Environments

Dynamic Distance Maintenance

Input graph G



Algorithm



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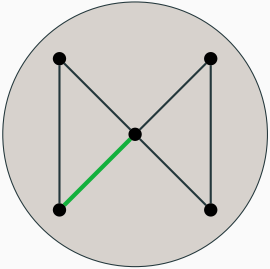

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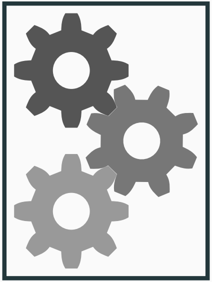

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Adversary inserts
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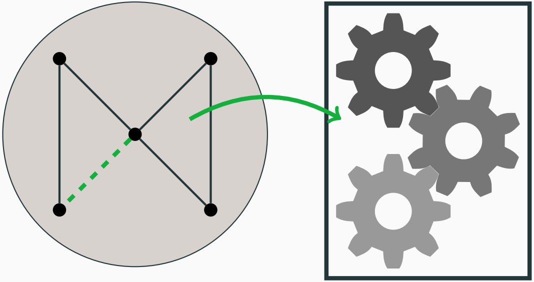

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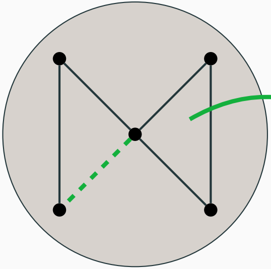

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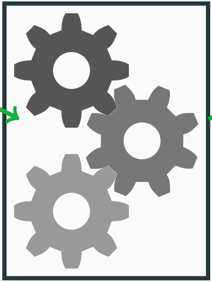

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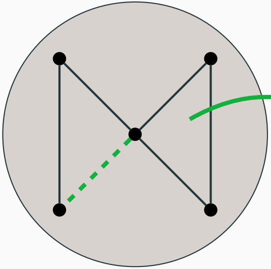

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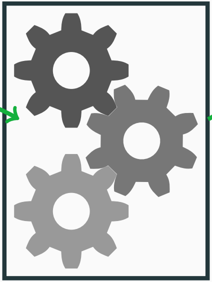

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State of the Art

Amortized update time $\tilde{O}(n^2)$ [Demetrescu, Italiano '03]

Subquadratic Update Time: State of the Art

- **Update-query time trade-offs:**
  - exact: [Sankowski ’05] [v.d. Brand, Nanongkai, Saranurak ’18]
  - $(1+\epsilon)$-approximation: [v.d. Brand, Nanongkai ’18]

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- **Large multiplicative stretch:**
  - Dynamic spanners: [Ausiello, Franciosa, Italiano ’05] [Elkin ’07] [Baswana, Khurana, Sarkar ’12], [Bodwin, K ’16] [Bernstein, F, Henzinger ’19] [Bernstein, v.d. Brand, Gutenberg, Nanongkai, Saranurak, Sidford, Sun ’22]
  - Dynamic distance oracles: [Abraham, Chechik, Talwar ’14] [F, Goranci, Henzinger ’21]

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- Fully dynamic
- Worst-case update time
- Deterministic
- Meet an update-time barrier

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Contribution

We add to this list: $(1+\epsilon)$-approximate distance approximation in unweighted, undirected graphs [van den Brand, **F**, Nazari arXiv ’21]

Our Results

Distance approximation in unweighted, undirected graphs:

Approx.	Type	Update Time
$1+\epsilon$	single pair	$O(n^{1.407})$
$1+\epsilon$	single source	$O(n^{1.529})$
$1+\epsilon$	k sources	$O(n^{1.529}+kn^{1+o(1)})$
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- Update times match (conditional) lower bounds [van den Brand, Nanongkai, Saranurak '19]

Further Results

Randomized Algorithms

- *Exact* single-pair distance: $O(n^{1.704})$
  (Improves upon $O(n^{1.724})$ [Sankowski '05] [v.d. Brand, Nanongkai, Saranurak '19])

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Warm Up

Randomized fully dynamic $(1 + \epsilon)$-approximate single-source distances with worst-case update time $O(n^{1.529})$.

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  - Return $\min(\hat{d}_G(s,v), d_H(s,v))$ for every node v

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Related Work

Randomized algorithm for maintaining $(1+\epsilon, n^{o(1)})$-spanner of size $n^{1+o(1)}$ with update time $O(n^{1.529})$ [Bergamaschi et al. '21]

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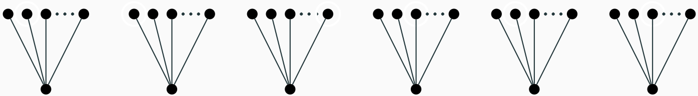

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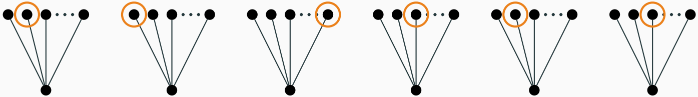

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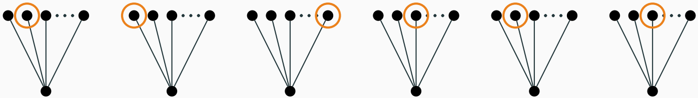

Adversarial model

Only works against an oblivious adversary

Emulator Construction

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A $(1+\epsilon, \beta)$-**emulator** of $G = (V, E)$ is a graph $H = (V, E')$ such that

$$dist_G(u, v) \leq dist_H(u, v) \leq (1+\epsilon) \cdot dist_G(u, v) + \beta$$

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Emulator H has two types of edges:

- For every light node of degree $\leq \sqrt{n}$: edges to all neighbors
- For every node in hitting set: (weighted) edges to all nodes in distance $\leq \lceil 6/\epsilon \rceil$

similar to [Henzinger, **K**, Nanongkai '13; Dor, Halperin, Zwick '97]

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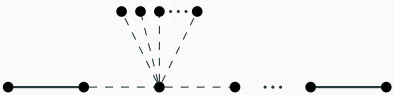

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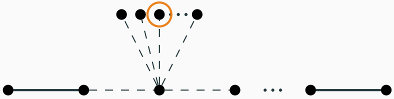

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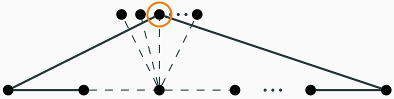

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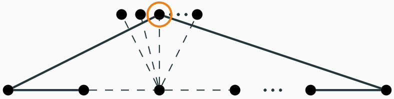

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  - If segment has length $\lceil 6/\epsilon \rceil$, then multiplicative error of $\leq \frac{\lceil 6/\epsilon \rceil + 2}{\lceil 6/\epsilon \rceil} \leq \frac{6/\epsilon + 3}{6/\epsilon} = 1 + \frac{\epsilon}{2}$

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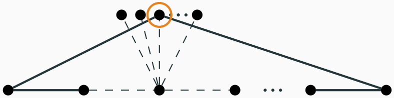

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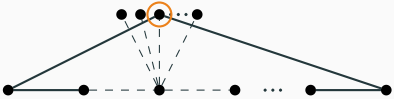

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Overall: multiplicative error of $1 + \frac{\epsilon}{2}$, additive error of 2

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Theorem ([Sankowski '05])

Given any $0 \leq \mu \leq 1$ and any sets $A, B \subseteq V$, there is a randomized data structure for maintaining the $A \times B$ distances up to $\leq h$ with update time $\tilde{O}((n^{\omega(1,\mu,1)-\mu} + n^{1+\mu} + |A| \cdot |B|) \cdot h)$.

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Approximation Guarantee:

- If $d_G(s, v) \leq \lceil 6/\epsilon \rceil$: distance from algebraic data structure

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- $O(n^{\omega(1,\mu,1)})$ denotes time needed for multiplying an $n \times n^{\mu}$ matrix with an $n^{\mu} \times n$ matrix
- With $\mu = 0.528\ldots$, update time is $\tilde{O}((n^{1.529} + |A| \cdot |B|) \cdot h)$
- With $A = S \cup \{s\}$, $B = V$ (where $|S| = \tilde{O}(\sqrt{n})$), and $h = O(1/\epsilon)$: update time $O(n^{1.529}/\epsilon)$

Approximation Guarantee:

- If $d_G(s, v) \leq \lceil 6/\epsilon \rceil$: distance from algebraic data structure
- If $d_G(s, v) > \lceil 6/\epsilon \rceil$, then approximation from H becomes $(1 + \frac{\epsilon}{2})d_G(s, v) + 2 \leq (1 + \frac{\epsilon}{2})d_G(s, v) + \frac{\epsilon}{3}d_G(s, v) \leq (1 + \epsilon)d_G(s, v)$

Towards Deterministic Algorithm

Three ideas:

1. Randomization not necessary in algebraic data structure for very small distances

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3. Algebraic data structure can be extended to slowly changing set of nodes

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Our approach:

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Dynamic Set Cover:

- Well studied problem [Gupta, Krishnaswamy, Panigrahi '17] [Abboud, Addanki, Grandoni, Panigrahi, Saha '19] [Bhattacharya, Henzinger, Nanongkai '19] [Bhattacharya, Henzinger, Nanongkai, Wu '21]
- Off-the shelf algorithms not applicable in our setting

Deterministic Path Counting

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Observation: For $k = O(1/\epsilon)$ we can live with overhead of $O(k^2 \log n) = \tilde{O}(1/\epsilon^2)$

Novel Algebraic Bounded-Distance Data Structure

Theorem

Given any $0 \leq \nu \leq \mu \leq 1$ *and any sets* $A, B \subseteq V$ *s.t.* $|A|, |B| \leq n^{\mu}$, *there is a randomized data structure for maintaining the* $A \times B$ *distances up to* $\leq h$ *under edge updates and* **set updates** *with update time* $\tilde{O}((n^{\omega(1,1,\mu)-\mu} + n^{\omega(1,\mu,\nu)-\nu} + n^{\mu+\nu} + |A| \cdot |B|) \cdot h^2)$.

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- $\tilde{O}((n^{1.407} + |A| \cdot |B|) \cdot h^2)$ for $|A|, |B| \leq n^{0.55}$
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- We extend approach of [v.d. Brand, Nanongkai, Saranurak '19] to optimize for case of large query set

Challenges

- “Path-reporting” for algebraic approaches [Bergamaschi, Henzinger, Gutenberg, Vassilevska Williams, Wein ’21] [Karczmarz, Mukherjee, Sankowski ’22]
- Extend emulator-based approximation approach to weighted graphs

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- Extend emulator-based approximation approach to weighted graphs
- More dynamic algorithms without caveats