An Update to Dynamic Graph Algorithms

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Paris Lodron University Salzburg

@ACSD 2024

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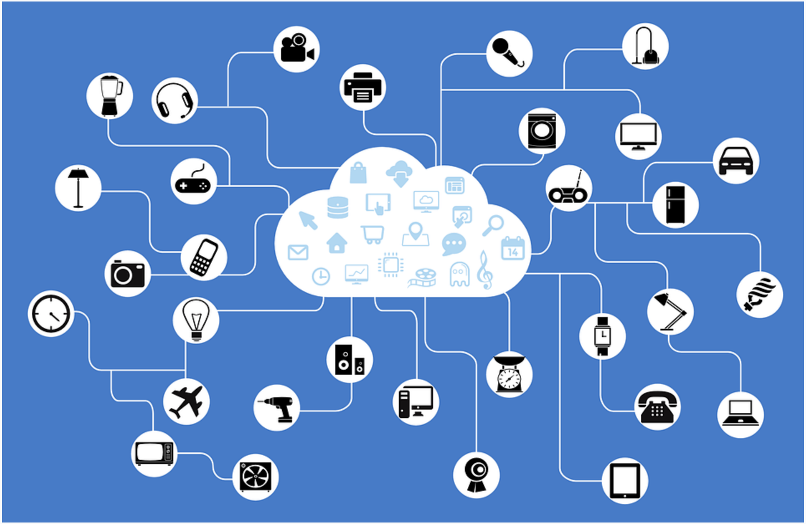
Graphs are Everywhere



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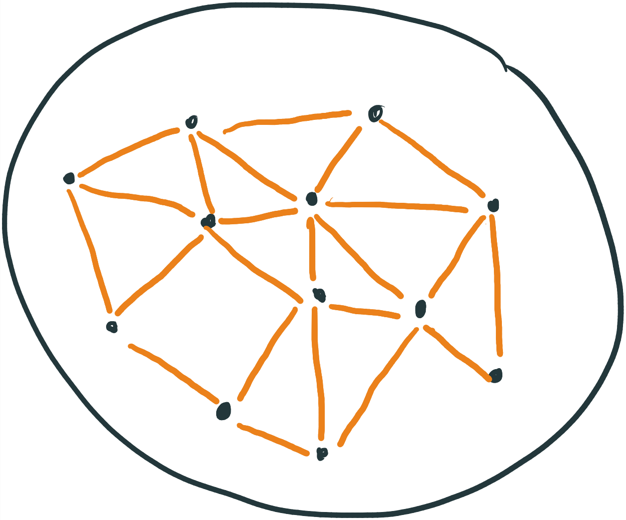
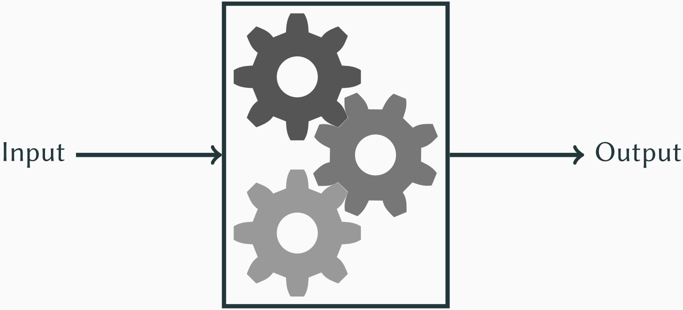


Static Algorithms



Dynamic Environments

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≈ 50 % of applications for big graphs are dynamic [Sahu et al. ’17]

Running Time

Goal

Design algorithms that react **quickly** to changes in the input data

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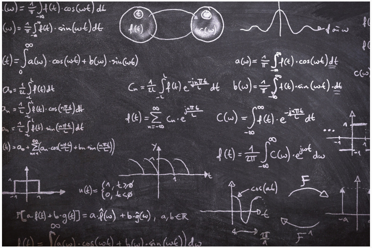
Measurement



Mathematical analysis

Warm-Up: Moving Average

Time series: $s_1, s_2, \dots, s_n$

Mean of last k values:

$\bar{s}_{n,k} = \frac{s_n + s_{n-1} + \dots + s_{n-k+1}}{k}$

$\rightarrow$ k arithmetic operations

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Efficiency gain!

Status Quo

Success Story:

- Fast dynamic graph algorithms for fundamental, “textbook” problems: connectivity, shortest paths, matching, ...
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Problem: (Too) little real-world impact

- Complicated algorithms
- Lack of (scalable) implementations
- Practitioners interested in wider array of problems

Towards Dynamic Graph Mining Algorithms

Vision

Systematically transfer technology developed for dynamic graph algorithms to graph mining and learning domain

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Focus on Relevant Problems:

- Centrality
- Clustering
- Pattern (subgraph) detection
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Recent survey [Hanauer, Henzinger, Schulz '22] reveals blind spots

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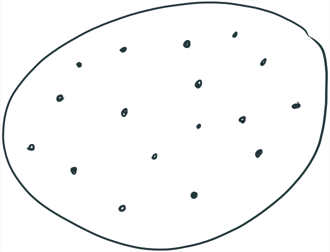
Integrated Pipeline

Algorithm design → Algorithm engineering → Applications

Case Study: k-Center Clustering

k-Center Problem

Given a metric space, select k points as set of centers C such that the maximum distance $d(C, v)$ of any node v to its closest center is minimized.



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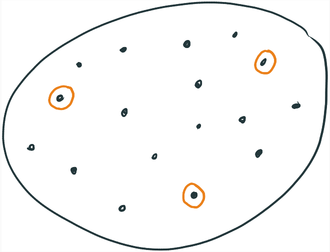

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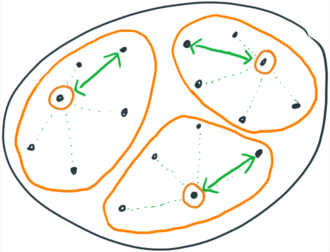

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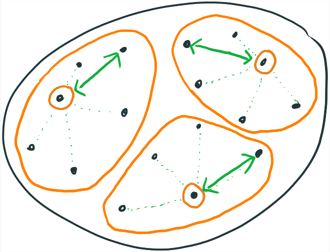

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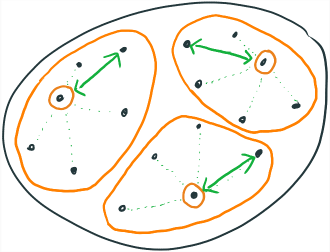
- Problem is NP-hard to approximate within a factor of $2 - \epsilon$
- Prior work for dynamic point sets [Chan, Gourqin, Sozio '18] [Bateni et al. '23]



Metric Spaces and Graphs

Definition (Metric on Point Set)

1. **Non-Negativity:** $d(x, y) \geq 0$
2. **Separation:** $d(x, y) = 0$ if and only if $x = y$
3. **Symmetry:** $d(x, y) = d(y, x)$
4. **Triangle Inequality:** $d(x, z) \leq d(x, y) + d(y, z)$

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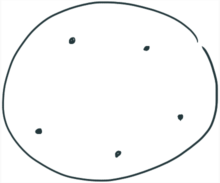
Question

Are there efficient dynamic constant-factor approximation algorithms for k-center if the metric is induced by a dynamically changing undirected graph?

Dynamic Model

Dynamic Point Sets:

- Point insertions and deletions



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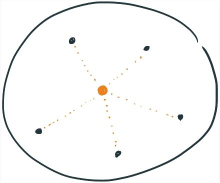
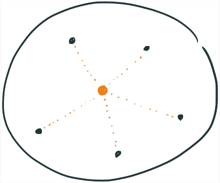


Dynamic Model

Dynamic Point Sets:

- Point insertions and deletions
- Query access to metric



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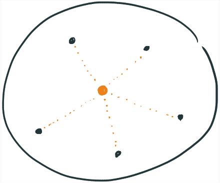

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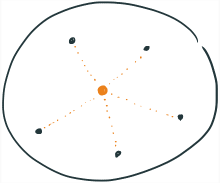
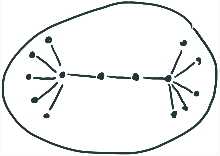


Dynamic Graphs:

- Edge insertions and deletions



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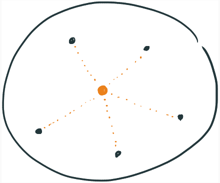

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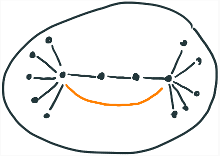

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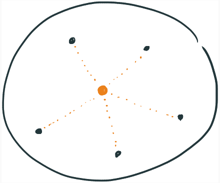
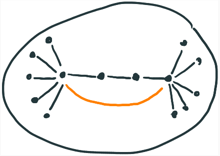


Dynamic Graphs:

- Edge insertions and deletions
- Distances not given for free



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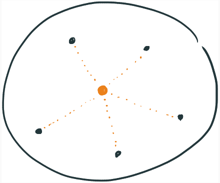

Dynamic Graphs:

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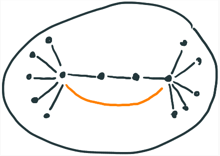

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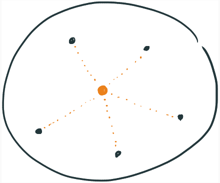

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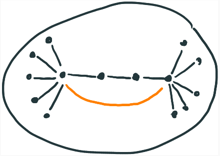

Conclusion

Cannot use results for dynamic point sets in a black-box manner for dynamic graph model

Related Work

Static Algorithms:

- Classic 2-approximation algorithms [Gonzalez '85] [Hochbaum, Shmoys '85]
  On graphs with n nodes and m edges: $\tilde{O}(km)$ time
- State of the art on graphs: $\tilde{O}(m)$ time (randomized) [Thorup '01] [Abboud et al. '23]

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Dynamic Point Sets:

- $\tilde{O}(k^2)$ update time [Chan, Gourqin, Sozio ’18]
- $\tilde{O}(k)$ update time [Bateni et al. ’23]
- Special cases: [Schmidt, Sohler ’19] [Goranci et al. ’21]
- Consistent k-center [Lattanzi and Vassilvitskii ’12] [Fichtenberger et al. ’21] [Łącki et al. ’23] [**F** and Skarlatos ’24]

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Natural goal: Update-time overhead of $\tilde{O}(k)$ compared to dynamic approximate single-source distances ("SSSP")

Our Results I: Fully Dynamic

Theorem (Cruciani, F, Goranci, Nazari, Skarlatos '24)

There is a fully dynamic $(2+\epsilon)$-approximate k-center algorithm with worst-case update time

- $O(kn^{1.529}\epsilon^{-2})$ *in unweighted graphs*
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Update time for fully dynamic $(1+\epsilon)$-approximate SSSP:

- $O(n^{1.529}\epsilon^{-2})$ (unweighted) [v. d. Brand, **F**, Nazari ’22]
- $O(n^{1.823}\epsilon^{-2})$ (weighted) [v. d. Brand, Nanongkai ’19]




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[implicit in Henzinger, **K**, Nanongkai ’14]

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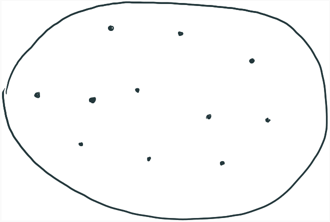
Gonzalez's Algorithm [Gonzalez '85]

1. Initialize $C = \{v\}$ with arbitrary first center
2. While $|C| < k$, add node v maximizing $d(C, v)$ to C



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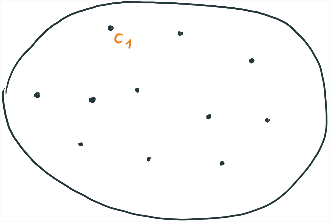

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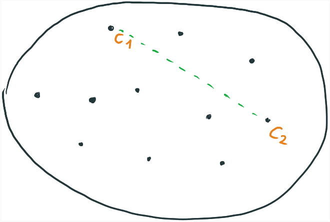

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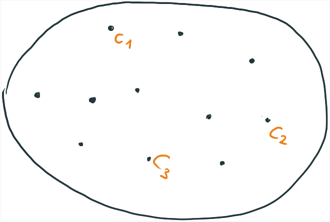

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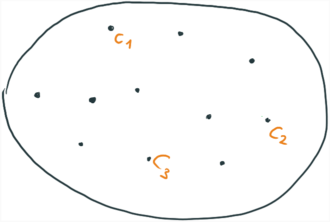

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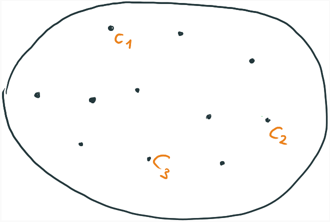

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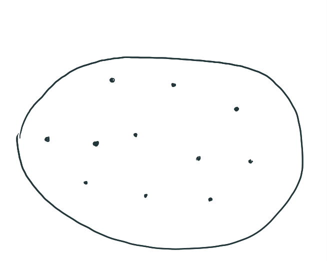
If $d(C, v)$ is within factor $1 + \epsilon$ of maximum, this gives $(2 + \epsilon)$-approximation

Fully Dynamic Algorithm: Simulating Gonzalez's Algorithm



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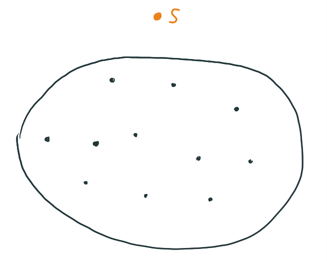
- Add artificial "super-source" s
- Maintain $(1 + \epsilon)$-approximate single-source distances from s with a fully dynamic algorithm



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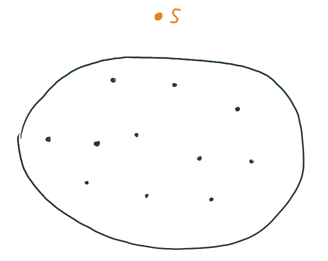

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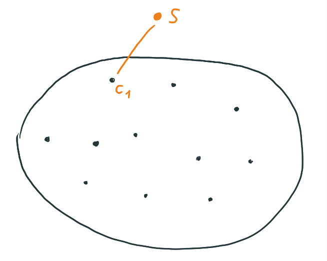

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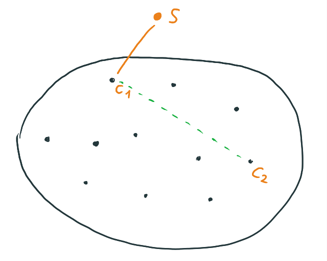

- After every update to graph:
  - Forward update to distance data structure
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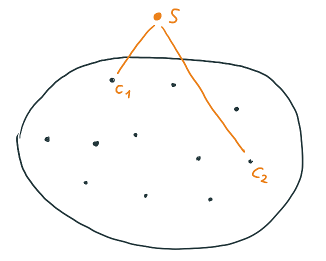

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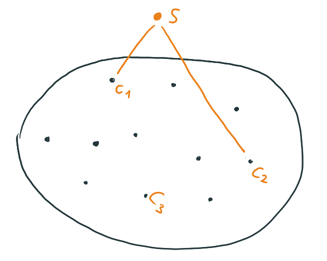

Fully Dynamic Algorithm: Simulating Gonzalez's Algorithm

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- After every update to graph:
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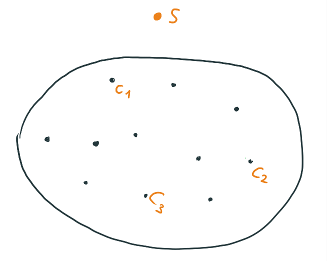

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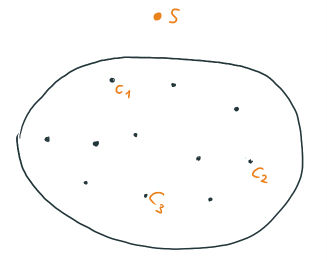

Update Time: $O(k \cdot U_{\text{SSSP}}(n))$

Outlook: Towards Dynamic Graph Mining Algorithms

Challenges:

- Experimental methodology not fully established
- Widespread use of heuristics in mining and learning domain
- Finding non-industrial applications

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Opportunities:

- Real-time data analysis
- Interesting research problems

Thanks for your attention!

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