# Near-Optimal Approximate Shortest Paths and Transshipment in Distributed and Streaming Models

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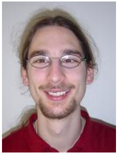
University of Vienna
→ University of Salzburg

joint work with



Ruben Becker
MPI Saarbrücken

Andreas Karrenbauer
MPI Saarbrücken

Christoph Lenzen
MPI Saarbrücken

# Approximate Single-Source Shortest Paths

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| **CONGEST** | $(\sqrt{n} + D) \cdot poly(\log n, \varepsilon)$ rounds |

1

2

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**Comments**:

- Undirected graphs with weights $\in \{1, 2, \ldots, poly(n)\}$
- $D$ = Diameter, $n$ = #nodes
- CONGEST lower bound: $\tilde{\Omega}(\sqrt{n} + Diam)$ rounds [Das Sarma et al '11]

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| **Streaming** | $poly(\log n, \varepsilon)$ passes $O(n \log n)$ space | $(2+1/\varepsilon)^{O(\sqrt{\log n \log\log n})}$ passes $O(n \log^2 n)$ space[3] |

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[3] [Elkin/Neiman '16]

# Approximate Single-Source Shortest Paths

| | Our $(1+\varepsilon)$-approx | Exact computation |
|---|---|---|
| **CONGEST** | $(\sqrt{n}+D)\cdot poly(\log n, \varepsilon)$ rounds | $n^{5/6} + D^{1/3}(n \log n)^{2/3}$ rounds[1] |
| **Cong. Clique** | $poly(\log n, \varepsilon)$ rounds | $O(n^{0.158})$ rounds[2] |
| **Streaming** | $poly(\log n, \varepsilon)$ passes $O(n \log n)$ space | $O(\frac{n}{k})$ passes $O(nk)$ space[3] |

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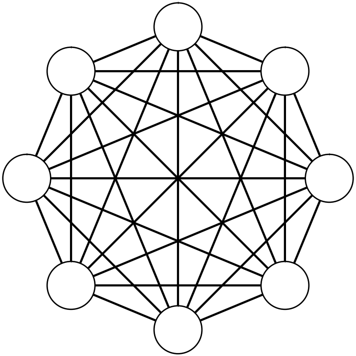
[2] [Censor-Hillel et al. '15]
[3] [Elkin '17]

# Broadcast Congested Clique



**Model:**

- Network topology: clique on $n$ nodes
- Synchronous rounds (global clock)
- In each round, every node sends one message to all other nodes
- Message size $O(\log n)$
- Local computation is free

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- Initially: Every node knows weight of its incident edges and whether it is the source or not
- Finally: Every node knows its approximate distance to the source

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**Simulation: Skeleton as congested clique** [Henzinger/K/Nanongkai '16]

$t$ rounds in Broadcast Congested Clique model $\rightarrow \tilde{O}(t \cdot (\sqrt{n} + Diam))$ rounds in CONGEST model

# Combinatorial Approach

# Sparsification I: Spanners

**Definition**

A $k$-spanner is a subgraph $H$ of $G$ such that, for all pairs of nodes $u$ and $v$, $dist_H(u, v) \leq k \cdot dist_G(u, v)$.

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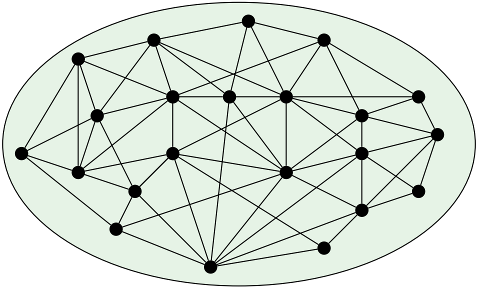

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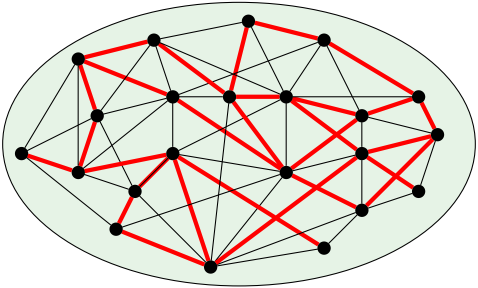

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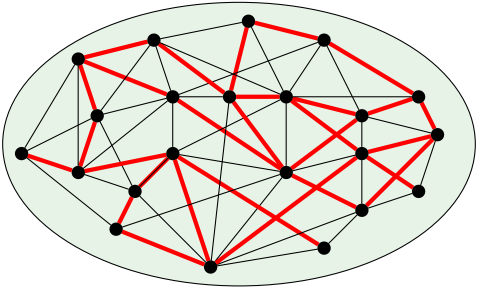

**Fact:** Every graph has a $(2k-1)$-spanner of size $n^{1+1/k}$

**Application:** Running time $T(m, n) \Rightarrow T(n^{1+1/k}, n)$

# Sparsification II: Hop Sets

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An $(h, \varepsilon)$-hop set is a set of weighted edges $F$ such that, for all pairs of nodes $u$ and $v$, in the 'shortcut graph' $G \cup F$ there is a path from $u$ to $v$ with **at most** $h$ **edges** of weight at most $(1 + \varepsilon)dist(u, v)$.

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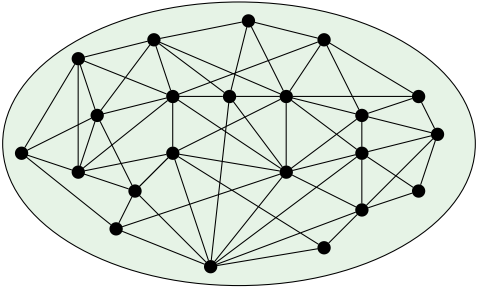

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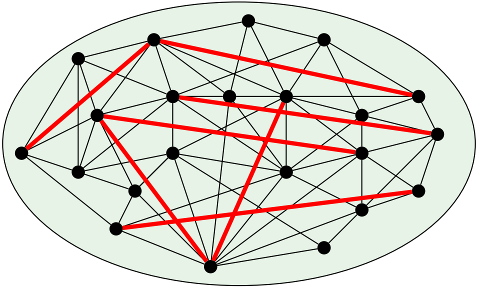

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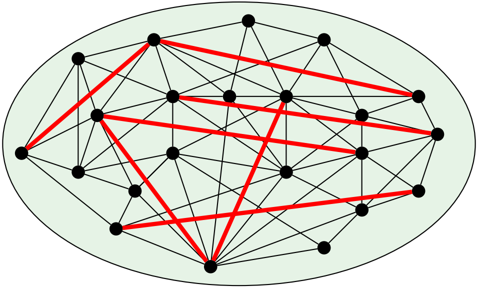

**Fact:** Every graph has a $(n^{o(1)}, \varepsilon)$-hop set of size $n^{1+o(1)}$ [Cohen '94] (for $\varepsilon \geq 1/polylogn$)

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**Application to approximate SSSP**

Almost tight algorithms for Bellman-Ford-like approaches:

- Parallel: $m^{1+o(1)}$ work with $n^{o(1)}$ depth [Cohen '94]
- Congested Clique: $n^{o(1)}$ rounds [Henzinger/K/Nanongkai '16]
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# Hop Sets: Approaching Optimality

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Hopset analysis of spanner/emulator in [Thorup/Zwick '06]

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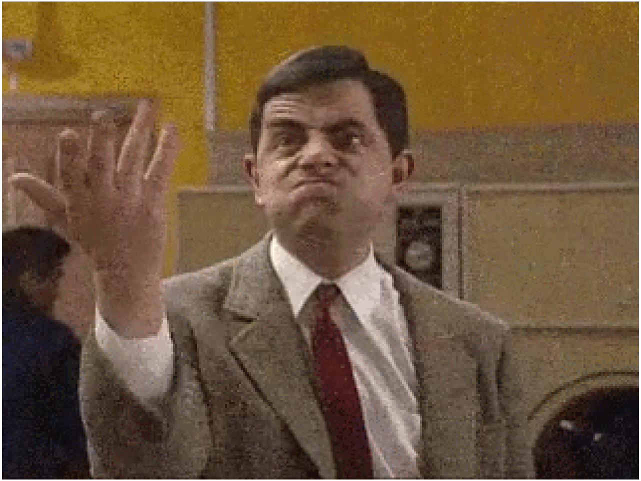
$\Rightarrow$ Cannot have $\alpha = 1+\varepsilon$, $h = poly(1/\varepsilon)$ and size $n \cdot polylog(n)$.

No further (significant) algorithmic improvements by better hop sets :(



It was too good to be true…

# Beyond Hop Sets

# Our Approach

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**Gradient Descent**

# Problem Formulation

## Shortest Transshipment Problem

Find the cheapest route for sending units of a single good from sources to sinks along the edges of a graph as specified by demands on nodes.

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## Flow View

Given demand $b(v)$ for each node $v$, find a flow $x(e)$ that:

- meets the demands: $\sum_{e=(u,v)\in E} x(e) = b(v) + \sum_{e=(v,u)\in E} x(e)$ for every node $v$
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Undirected graphs: arbitrary orientation of edges.

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**SSSP:** source has demand $-(n-1)$, other nodes have demand 1

# Reformulation

## LP Formulation

**Primal:** minimize $\|Wx\|_1$ s.t. $Ax = b$

**Dual:** maximize $b^T y$ s.t. $\left\|W^{-1}A^T y\right\|_\infty \leq 1$

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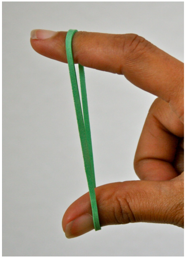

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$|y(u) - y(v)|/w(e) \leq 1$ for every edge $e = (u, v)$
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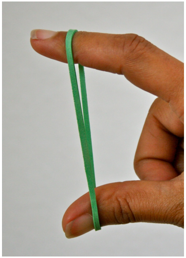

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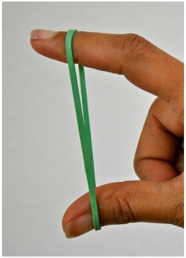

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We approximate $\|\cdot\|_\infty$ by soft-max:

$$\mathsf{lse}_\beta(x) := \frac{1}{\beta} \ln\left(\sum_{1\leq i\leq n}\left(e^{\beta x_i} + e^{-\beta x_i}\right)\right)$$

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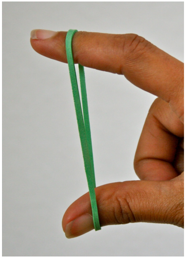

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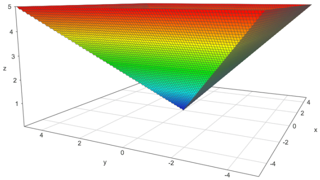
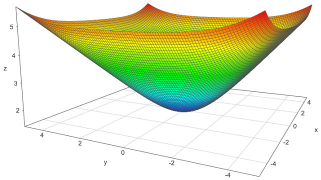
**Goal:** minimize $\Phi_\beta(\pi) := \mathsf{lse}_\beta(W^{-1}A^T \pi)$ s.t. $b^T \pi = 1$

# Soft-max approximation




$\Rightarrow$

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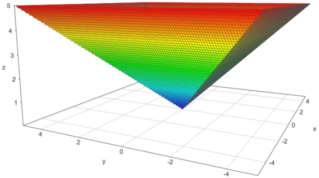


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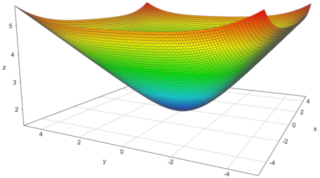


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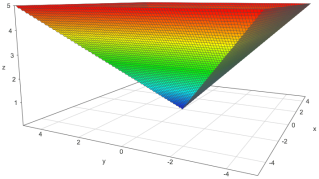
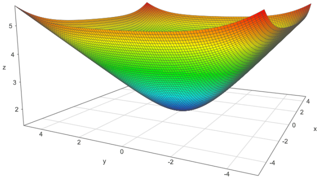



$$\Longrightarrow$$

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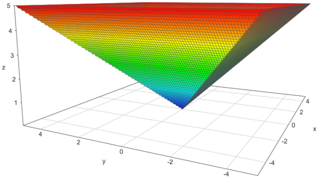

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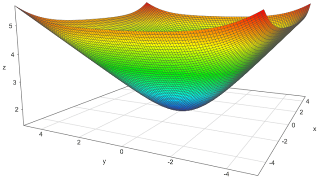

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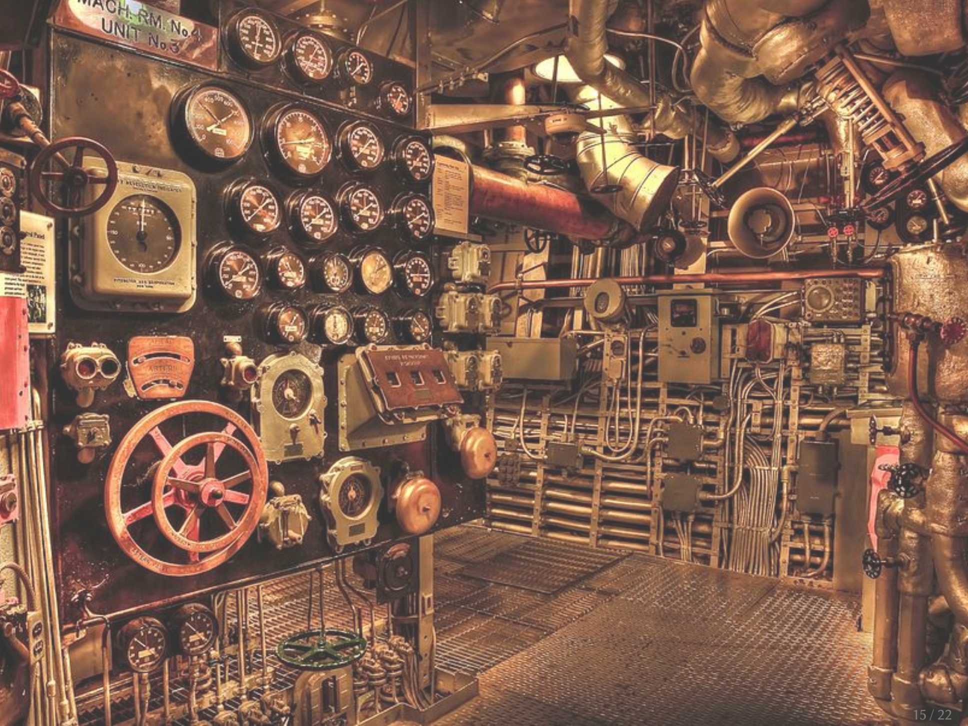
**Intuition:** Trade off quality of approximation and smoothness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UNIT No.3

# Generic Update Step

Bounding improvement in objective for generic update $\pi' = \pi - h$:

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$$\Phi_\beta(\pi) - \Phi_\beta(\pi - h)$$
$$\geq \nabla\Phi_\beta(\pi - h)^T h$$

**Convexity:** $f(y) \geq f(x) + \nabla f(x)^T (y - x)$

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**Chain rule:** $\nabla\Phi_\beta(\pi) = AW^{-1}\nabla \operatorname{lse}_\beta(W^{-1}A^T\pi)$

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**Hölder:** $x^T y \leq \|x\|_p \|y\|_q$ for $\frac{1}{p} + \frac{1}{q} = 1$

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**Lipschitz:** $\left\|\nabla \operatorname{lse}_\beta(x) - \nabla \operatorname{lse}_\beta(y)\right\|_1 \leq \beta \|x - y\|_\infty$

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$\Rightarrow$ Solve on spanner with stretch $\alpha = \log n$ of size $O(n \log n)$ ("oracle")

# Gradient Descent Algorithm

**repeat**

  **while** $\frac{4\ln(4m)}{\varepsilon\beta} \geq \Phi_\beta(\pi)$ **do** $\beta \leftarrow \frac{5}{4}\beta$.

  $\tilde{b} \leftarrow P^T \nabla\Phi_\beta(\pi)$, where $P \leftarrow I - \pi b^T$

  $\tilde{h} \leftarrow \alpha$-approximation of $\max\{\tilde{b}^T h : \left\|W^{-1}A^T h\right\|_\infty \leq 1\}$

  $\delta \leftarrow \frac{\tilde{b}^T\tilde{h}}{\left\|W^{-1}A^T P\tilde{h}\right\|_\infty}$

  **if** $\delta > \frac{\varepsilon}{8\alpha}$ **then** $\pi \leftarrow \pi - \frac{\delta}{2\beta\left\|W^{-1}A^T P\tilde{h}\right\|_\infty} P\tilde{h}$.

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## Theorem

*Given an $\alpha$-approximate shortest transshipment oracle, one can compute primal solution $x$ and dual solution $y$ such that $\|Wx\|_1 \leq (1+\varepsilon)b^T y$ with $(\varepsilon^{-3}\alpha^2 \log n \log \alpha)$ oracle calls.*

# Implementation in Brodcast Congested Clique

1. **Evaluate Gradient:**
   - Evaluate $(\nabla\Phi_\beta(\pi))_v$ locally at each node $v$
   - $(\nabla\Phi_\beta(\pi))_v$ is a function of $\pi$ and weight of edges incident to $v$ ("edge stretches under current node potentials")
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2. **Oracle call:**
   - Initially compute spanner in $O(\log n)$ rounds [Baswana/Sen '03]
   - Spanner then is global knowledge (size $O(n \log n)$)
   - At oracle call, make gradient global knowledge (size $O(n)$)
   - Each node can internally compute solution on spanner

# Are we done?

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**Theorem**

*We can compute a $(1+\varepsilon)$-approximate distance estimate for each node in the SSSP problem with polylog$(n, \|w\|_\infty)$ calls to our gradient descent algorithm with precision $\varepsilon' = \Omega(\varepsilon^3/(\alpha^2 \log n))$.*

# Comparison to [Sherman SODA'17]

Both papers solve $(1 + \varepsilon)$-approximate shortest transshipment

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  $\Rightarrow$ nearly tight approximate SSSP in distributed and streaming models

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2. **Parallel Model:** Approximate SSSP with $m \cdot poly(\log n, \varepsilon)$ work and $poly(\log n, \varepsilon)$ depth?
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**Thank you!**