Advances in Fully Dynamic Algorithms with Worst-Case Update Time Guarantees

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Max Planck Institute for Informatics
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Dagstuhl Seminar
"Structure and Hardness in P"

Our world is not static

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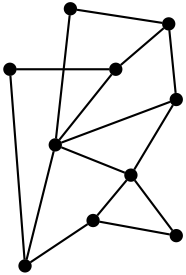
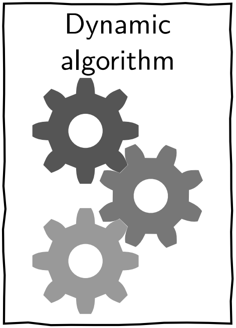
Dynamic model

G undergoing updates:





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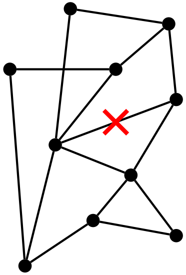

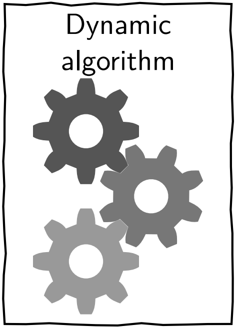

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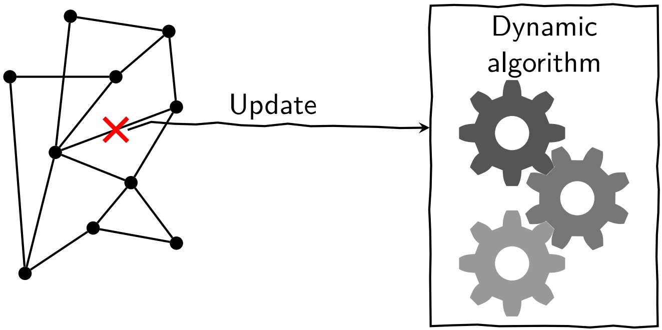

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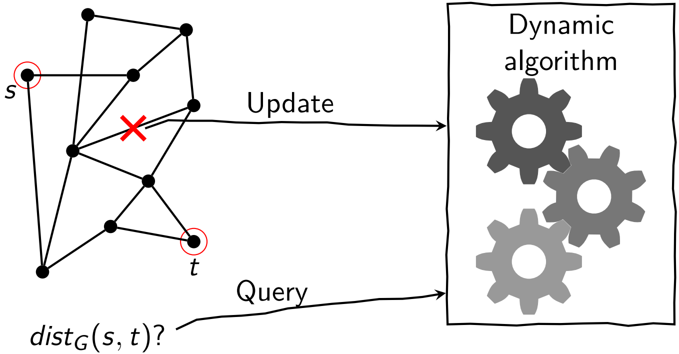

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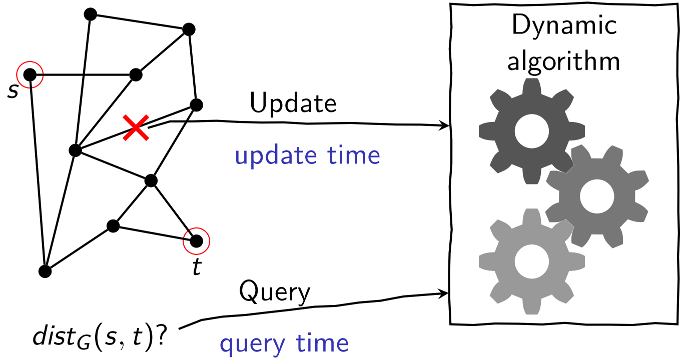

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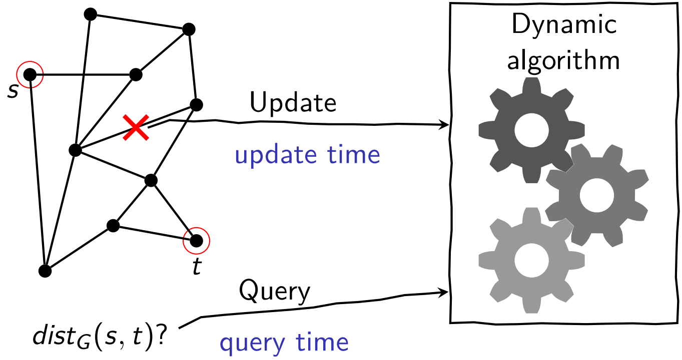

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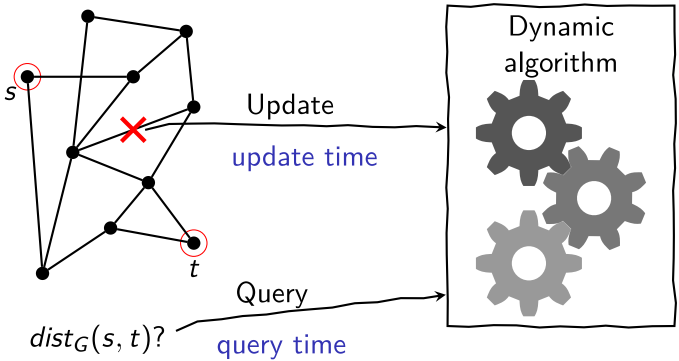

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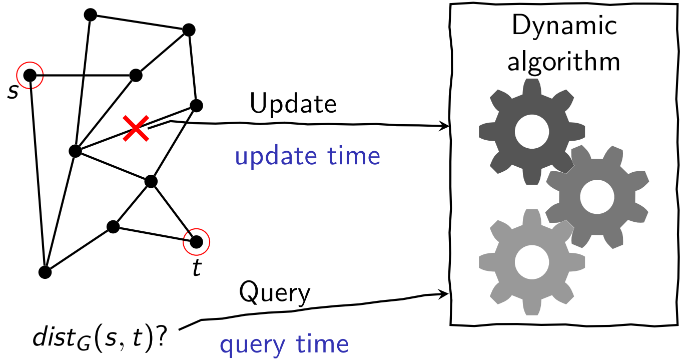

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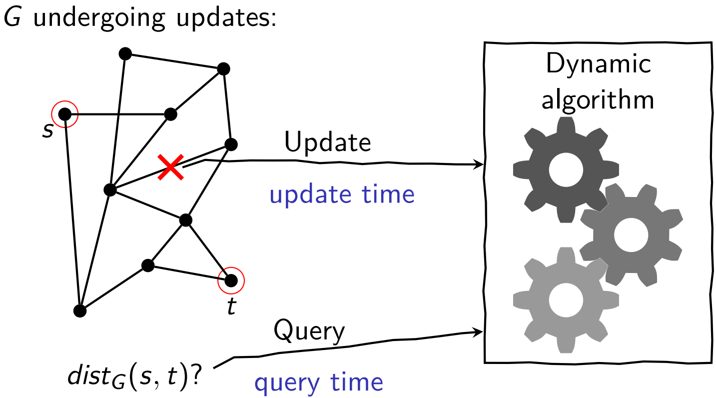

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Goal: Minimize update time $T(n, m)$

- Worst-case: After each update, spend time $\leq T(n, m)$
- Amortized: For a sequence of k updates, spend time $\leq kT(n, m)$

Why worst-case bounds?

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Be punctual!

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Sink of real-time systems!

Overview

	amortized	**worst-case**
Connectivity	$\log^{\mathcal{O}(1)} n$ [Henzinger/King '95]	$\log^{\mathcal{O}(1)} n$ [Kapron/King/Mountjoy '13]

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$(1+\epsilon)$-cut sparsifier	$\log^{\mathcal{O}(1)} n$ [Abr./Durfee/Koutis/K/Peng '16]	$\log^{\mathcal{O}(1)} n$ [Abr./Durfee/Koutis/K/Peng '16]

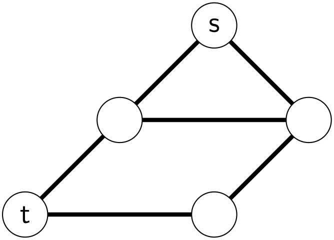
Question:

Can worst-case bounds match amortized bounds?

Case study: APSP

Joint work with Ittai Abraham and Shiri Chechik

Simple example – Distance from s to v



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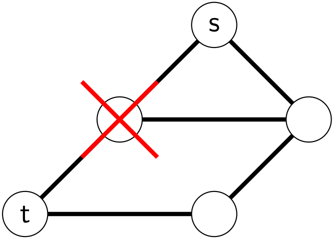

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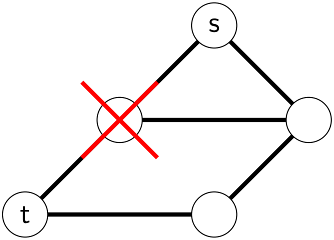

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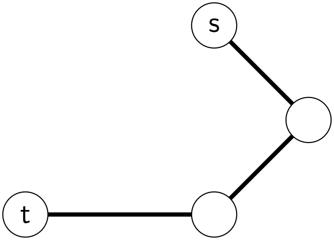
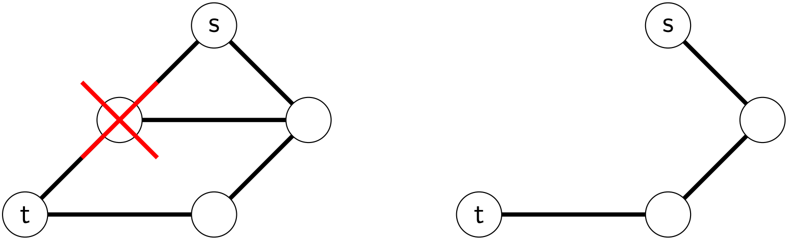


dist(s, t) $= 3$

Simple example – Distance from s to v



dist(s, t) $=$ 2   dist(s, t) $=$ 3

Dynamic shortest paths data structure:

- initialize(G)
- insert(v)
- delete(v)
- dist(s, t)
- path(s, t)

$\left.\begin{array}{l}\text{insert}(v)\\ \text{delete}(v)\end{array}\right\}$ update

$\left.\begin{array}{l}\text{dist}(s,\ t)\\ \text{path}(s,\ t)\end{array}\right\}$ query

Prior work

approx.	update time	type of graphs	reference
exact	$\tilde{\mathcal{O}}(mn)$	weighted directed	[Dijkstra]
exact	$\tilde{\mathcal{O}}(n^{2.5}\sqrt{W})$	weighted directed	[King '99]
$1+\epsilon$	$\tilde{\mathcal{O}}(n^2 \log W)$	weighted directed	[King '99]
$2+\epsilon$	$\tilde{\mathcal{O}}(n^2)$	weighted directed	[King '99]
exact	$\tilde{\mathcal{O}}(n^{2.5}\sqrt{W})$	weighted directed	[Demetrescu/Italiano '01]
exact	$\tilde{\mathcal{O}}(n^2)$	weighted directed	[Demetrescu/Italiano '03]
exact	$\tilde{\mathcal{O}}(n^{2.75})$ (*)	weighted directed	[Thorup '05]
$2+\epsilon$	$\tilde{\mathcal{O}}(m \log W)$	weighted undirected	[Bernstein '09]
$2^{\mathcal{O}(k)}$	$\tilde{\mathcal{O}}(\sqrt{m}n^{1/k})$	unweighted undirected	[A/C/Talwar '14]

(*) worst case

$\tilde{\mathcal{O}}$: ignores $\log n$-factors
n: number of nodes
m: number of edges
W: largest edge weight

Our result

Theorem (for this talk)

There is an algorithm for maintaining a distance matrix under insertions and deletions of nodes in unweighted undirected graphs with a worst-case update time of $\tilde{\mathcal{O}}(n^{2.75})$.

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Toy example! ($O(n^{\omega})$ in unweighted graphs)

Our result

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Toy example! ($O(n^{\omega})$ in unweighted graphs)

More sophisticated use of our technique:

- $\tilde{\mathcal{O}}(n^{2.67})$ in weighted directed graphs
- Improves $\tilde{\mathcal{O}}(n^{2.75})$ of [Thorup '05]
- (Hopefully) simpler than [Thorup '05]
  (which is a deamortization of [Demetrescu/Italiano '03])

Restrictions

Known techniques allow the following restrictions:

1. Only necessary to maintain shortest paths up to length h (for some parameter h)

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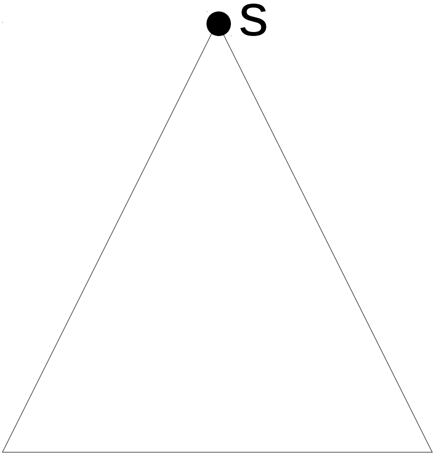
Known techniques allow the following restrictions:

1. Only necessary to maintain shortest paths up to length h (for some parameter h)
2. To obtain a fully dynamic algorithm it is sufficient to design a deletions-only algorithm that
   - can handle up to Δ deletions of nodes with worst-case guarantees
   - after **preprocessing** the graph

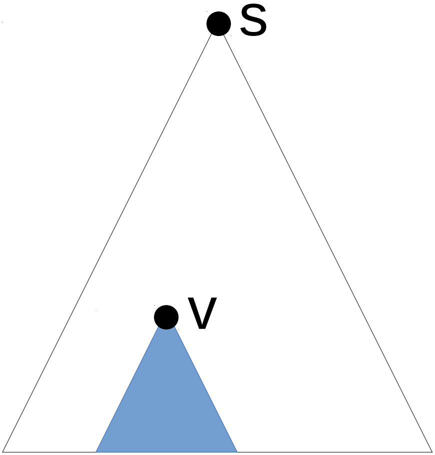
   Restart deletions-only algorithm each Δ updates
   (Preprocessing time can be amortized over previous Δ deletions!)
   Floyd-Warshall to process Δ insertions in time $\mathcal{O}(\Delta n^2)$

Repairing a shortest path tree



- Given: shortest path tree from s

Repairing a shortest path tree



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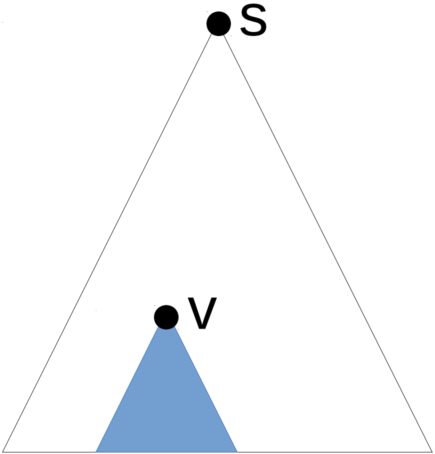
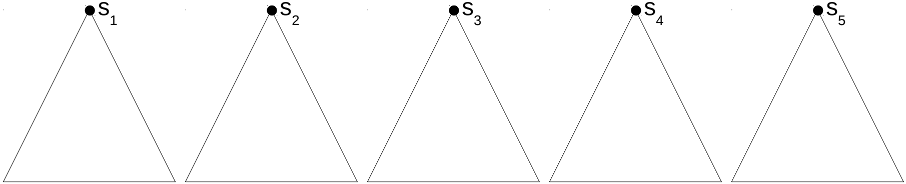


- Given: shortest path tree from s
- Node v is deleted
- Shortest path destroyed only for nodes in subtree of v
- Run Dijkstra's algorithm to reattach these nodes to the tree
- Charge time $\mathcal{O}(deg(u)) \leq \mathcal{O}(n)$ to every node u in subtree of v

Multiple shortest path trees

Goal: shortest paths from a set of source nodes S



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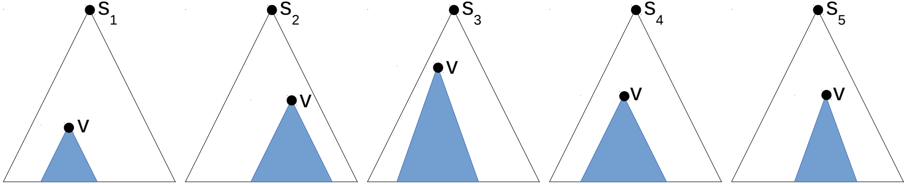

Deletion of v

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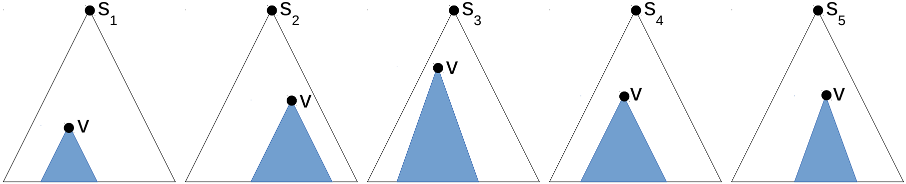

Deletion of v

Total work: (number of nodes in subtrees of v) $\times n$

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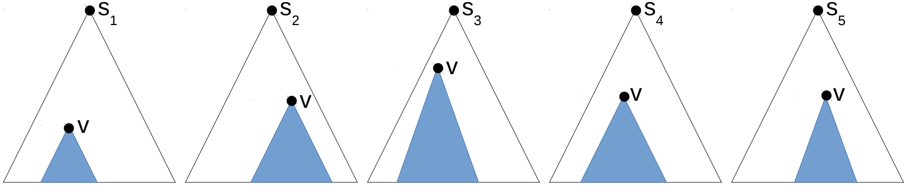

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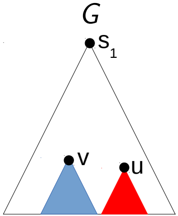
Goal: limit sizes of subtrees of each node

Preprocessing

For every source, construct shortest path tree up to depth h:



Count size of subtrees for every node

Rule: If number of nodes in subtrees of v exceeds λ:

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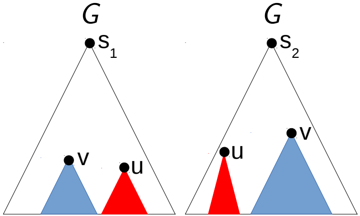

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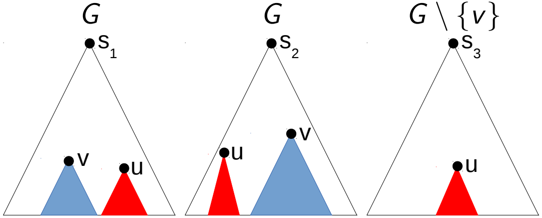

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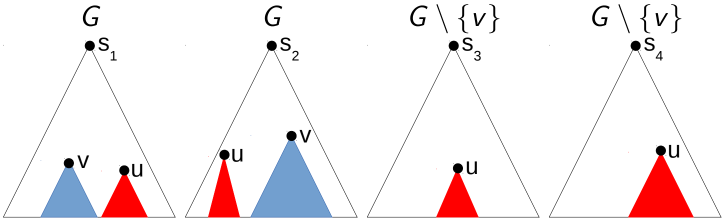

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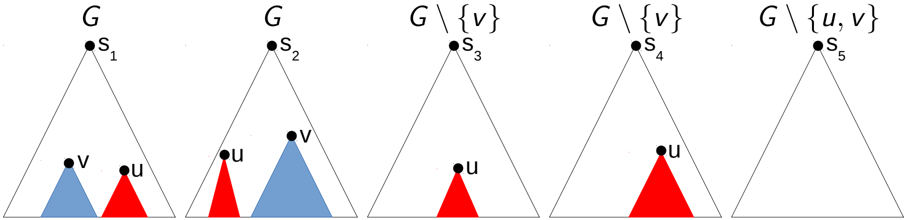

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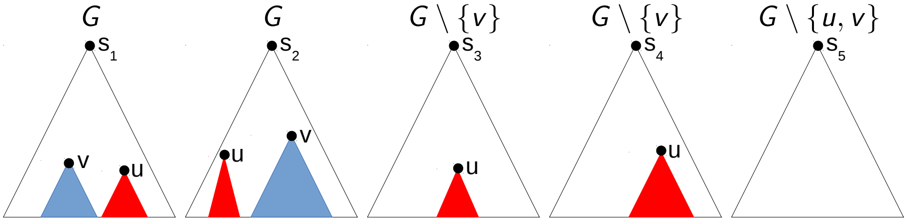

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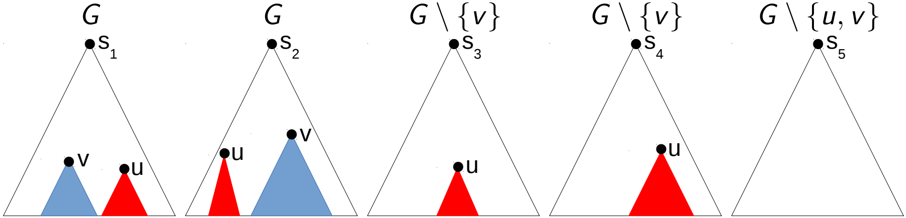
Observations:

- All shortest paths not using heavy nodes included in trees
- Number of heavy nodes: $|H| \le \mathcal{O}(\frac{|S|nh}{\lambda}) \le \mathcal{O}(\frac{n^2 h}{\lambda})$
- Preprocessing time: $\mathcal{O}(|S|n^2) \le \mathcal{O}(n^3)$

Computing distances after deletions



1. For all deleted nodes: Reattach children to tree using Dijkstra
   Running time: $\mathcal{O}(\Delta\lambda n)$ per deletion
   - Subtree size at most λ per node
   - Number of deleted nodes at most Δ

   Correct for all shortest paths not containing heavy nodes

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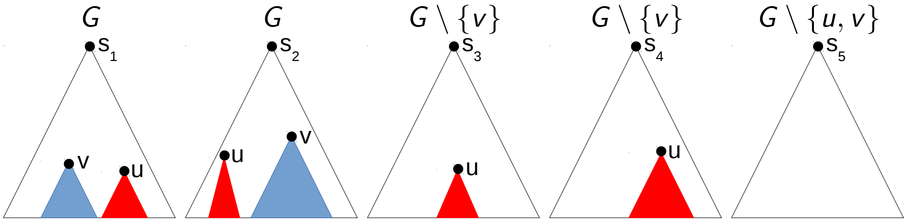

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2. Special treatment of heavy nodes: shortest paths via heavy nodes

   $$\text{Compute } \min_{v \in H}(dist(s, v) + dist(v, t)) \text{ for all } s \text{ and } t$$

   Time per deletion: $\mathcal{O}(|H|n^2) = \mathcal{O}(\frac{n^4 h}{\lambda})$

Running time wrapped up

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$$\Delta = n^{0.25},\ \lambda = n^{1.5},\ h = n^{0.25}$$
$$\Rightarrow \tilde{\mathcal{O}}(n^{2.75})$$

Improvements

Directed graphs:
Two types of shortest path trees: incoming and outgoing

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Two types of shortest path trees: incoming and outgoing

Weighted graphs:
Length of path $\rightarrow$ number of nodes on path ("hops")
Requires Bellman-Ford in preprocessing: $O(n^2 h)$ per node

Improvements

Directed graphs:
Two types of shortest path trees: incoming and outgoing

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Length of path $\rightarrow$ number of nodes on path ("hops")
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Increased efficiency:
Multiple instances of algorithm to cover all hop ranges (+randomization)
Load balancing trick

Barriers

Combinatorial approach [Thorup '05, Abraham/Chechik/Krinninger '17]

The best we can hope for:

- Preprocessing: $\mathcal{O}(n^3)$
- "Amortize" preprocessing over k updates: $\mathcal{O}(n^3/k)$
- Deal with $\leq k$ insertions after each update: $\mathcal{O}(n^2 k)$

$\Rightarrow \mathcal{O}(n^{2.5})$

Algebraic approach [Sankowski '04/'05]

Here: Intuition in DAGs

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Transitive closure:

- Count number of paths from s to t for all pairs
- Reachable iff $\#\text{paths} > 0$
- Perform operations for counting modulo random prime
- Update time $\mathcal{O}(n^2)$
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All-pairs shortest paths (distances):

- For every $1 \leq \ell \leq h$, count #paths of length exactly ℓ
- Additional trick: fast convolution
- Update time: $\tilde{\mathcal{O}}(n^2 h)$.
- Standard trick for hitting long paths: $h = \sqrt{n}$

$\Rightarrow \mathcal{O}(n^{2.5})$

Conclusion

Worst-case update time is the real deal![1]

[1]. . . and correctness against adaptive online adversary $\rightarrow$ Thatchapol Saranurak's talk

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- Often requires new approaches
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Conclusion

Worst-case update time is the real deal![1]

- Often requires new approaches
- Technically challenging
- Conditional lower bounds: No technique yet to separate amortized and worst-case update time for **fully dynamic** problems

[1] ... and correctness against adaptive online adversary $\rightarrow$ Thatchapol Saranurak's talk

Open problems

How about the following worst-case bounds?

- Fully dynamic APSP: Meet $n^{2.5}$ barrier
- Fully dynamic APSP: $(1+\epsilon)$-approximation in $\tilde{\mathcal{O}}(n^2/\epsilon)$ time?
- Fully dynamic transitive closure: deterministic $\mathcal{O}(n^2)$ algorithm?