Deterministic Incremental APSP with Polylogarithmic Update Time and Stretch

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Joint work with Yasamin Nazari and Maximilian Probst Gutenberg

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Our Result

Theorem

There is a deterministic algorithm that, given an undirected graph with real edge weights in $[1, W]$ *undergoing edge insertions, maintains in total time* $O(m \log n \log\log n + n \log^6(nW) \log\log n)$ *over all updates a distance oracle with polylogarithmic stretch and query time* $O(\log\log n)$*, where* n *denotes the number of vertices and* m *denotes the final number of edges of the graph.*

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Conditional lower bound

Constant stretch likely needs polynomial update time [Abboud, Bringmann, Khoury, Zamir '22, Abboud, Bringmann, Fischer '22]

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Notable:

- No Even-Shiloach tree
- No expander decomposition
- No Thorup-Zwick based construction

$n^{o(1)}$ Barrier

Decremental:

- [Chechik '18] (rand.) stretch $O(\log n)$, total time $mn^{o(1)}$
- [Łącki, Nazari '22] (rand.) stretch $O(\log n)$, total time $\tilde{O}(m + n^{1+o(1)})$
- [Chuzhoy '21] (det.) polylog. stretch, total time $O(m^{1+\delta})$
- [Bernstein, Chechik] (det.) $(1+\epsilon)$-approx. SSSP, total time $\tilde{O}(n^2)$.

Incremental:

- [Chen, Goranci, Henzinger, Peng, Saranurak '20] (det.) stretch $O(1)$, total time $O(m^{1+o(1)})$

A Case for Partially Dynamic Algorithms

Decremental:

- Hope: extend to fully dynamic by reductions
- Useful for static algorithms
  - multicommodity flow [Mądry ’10,Chuzhoy ’21]
  - (approximate) min-cost flow [Bernstein, Probst Gutenberg, Saranurak ’21, Chen, Kyng, Liu, Peng, Probst Gutenberg, Sachdeva ’22]
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Incremental:

- Natural growth processes (co-authors, Wikipedia links, ...)
- Search for implementable algorithms

Static Construction: Hierarchy

[Andoni, Stein, Zhong '20]

Overall setup:

- Hierarchy of $k = \Theta(\log\log n)$ sparsifiers: $G = H_1, H_2, \ldots, H_k$
- $|V(H_{i+1})| = |V(H_i)|/b_i$ for double exponentially increasing b_i's

$$|V(H_i)| = O\left(\frac{n}{b_1 \cdot b_2 \cdot \cdots \cdot b_{i-1}}\right)$$

$$|E(H_i)| \leq m + O\left(\frac{n}{b_1 \cdot b_2 \cdot \cdots \cdot b_{i-1}} \cdot b_i\right)$$

- H_i is an α-approximation of H_{i-1} for some constant α
  $\alpha^k = \operatorname{polylog} n$

Static Construction: One Level

- Nodes of H_{i+1}: Randomized hitting set of size $\tilde{O}(n/b_i)$
- Compute b_i-ball around each node (b_i closest nodes)
  b_i-ball of u contains sampled node $p_i(u)$ ("pivot")

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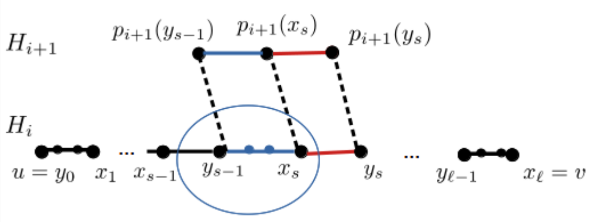
- $|E(H_{i+1})| \leq |E(H_i)| + |V(H_i)| \cdot b_i$



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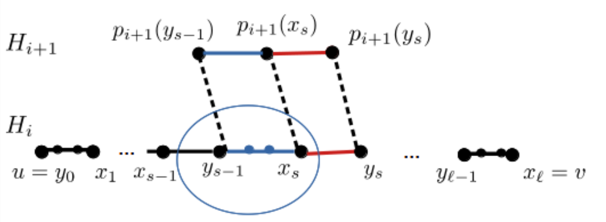

segment $y_{s-1} \to y_s$ approximated in H_{i+1} with multiplicative stretch α and additive stretch $d_{H_i}(y_s, p_i(y_s))$

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Deterministic maintenance of pivots:

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Deterministic maintenance of pivots:

- Dynamic version of Greedy
- Charging scheme: each pivot reduces approximate pivot distance of b_i nodes by a constant factor
  In total: $\tilde{O}(|V(H_i)|/b_i)$ pivots $(= |V(H_{i+1})|)$


The
End


GAME
OVER

The problem:

- Incremental algorithm $\mathscr{A}_i$ maintaining H_{i+1} based on H_i
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- Number of edges added to H_{i+1}: $O(|V(H_i)|b_i^2 \log n + |E(H_i)| \log n)$
- For $k = \Theta(\log\log n)$ levels: $m \cdot O(\log n)^{\log\log n}$ insertions (at best)
- Will not give $O(m \operatorname{polylog} n)$ total update time

Solutions and More Challenges

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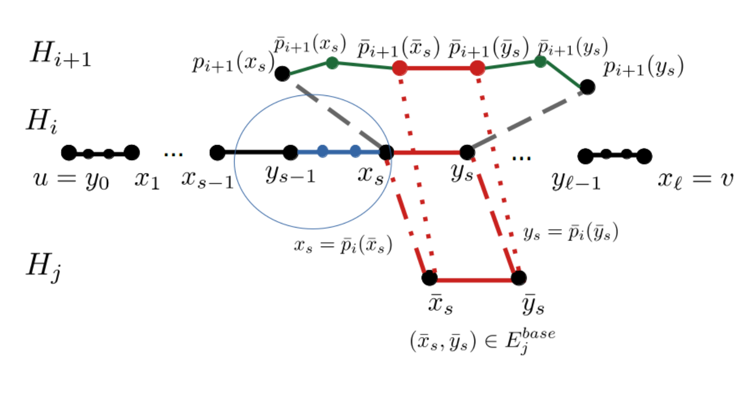
Challenge:

- We've opened up the "black box"
- But: level-by-level analysis was very convenient for correctness proof
- Problem: pivots might be out of sync with actual projections

The Full Picture



Can make it work by maintaining additional types of edges

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- Would expect $O(\log n)$ changes of pivot in that range

But: cannot simply maintain SSSP from set of pivots

(standard approach in Thorup-Zwick based constructions)

Questions

Decremental?

Worst-case update time?