Recent Results on Dynamic Distance Computation

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Joint works with Jan van den Brand, Michal Dory, Gramoz Goranci, Yasamin Nazari, Maximilian Probst Gutenberg, Antonis Skarlatos, and Tijn de Vos

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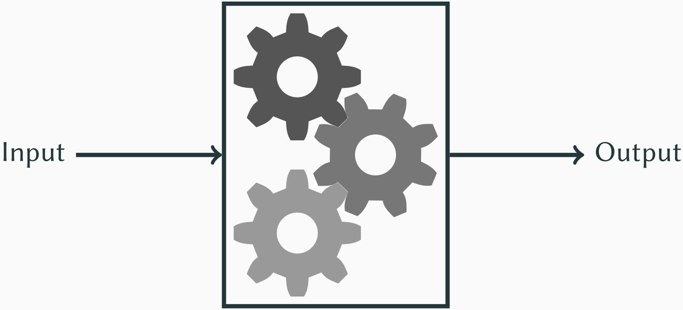
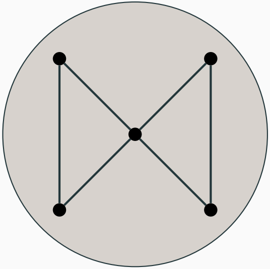
Static Approach



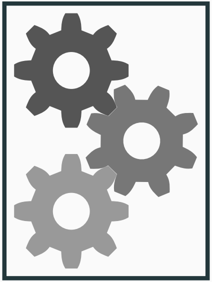
Dynamic Environments

Dynamic Distance Maintenance

Input graph G



Algorithm



Distance Matrix

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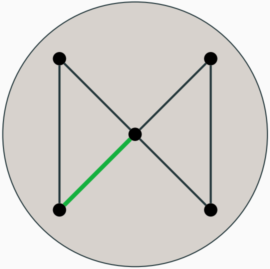
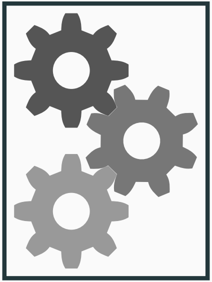


Adversary inserts and deletes edges

Algorithm



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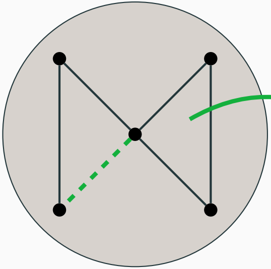

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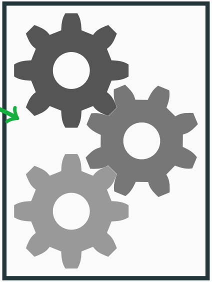

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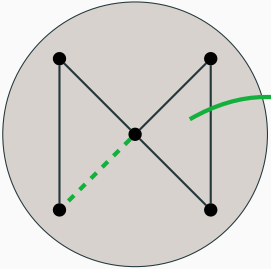

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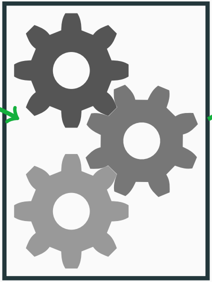

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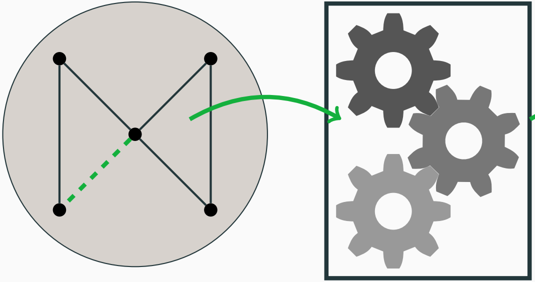
Input graph G — Algorithm — Distance Matrix



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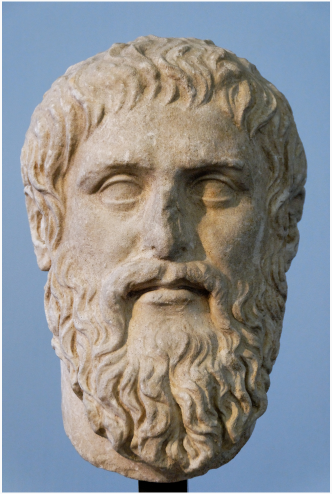
State of the Art

Amortized update time $\tilde{O}(n^2)$ [Demetrescu, Italiano ’03]

1. Exact
2. $1 + \epsilon$
3. Small constant (2 to < 3)
4. Thorup-Zwick (3 to $O(\log n)$)
5. Above Logarithmic



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Fully dynamic:

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- Deterministic worst-case update time:
  - $\tilde{O}(n^{2+41/61})$ [Chechik, Zhang ’23]
  - $\tilde{O}(n^{2.6})$ unweighted [Probst Gutenberg, Wulff-Nilsen ’20]
- Trade-off: worst-case update time/query time $O(n^{1.724})$
  [Sankowsi ’05; v.d. Brand, Nanongkai, Saranurak ’19]

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Decremental:

- Total update time $\tilde{O}(n^3)$ in unweighted graphs [Demetrescu, Italiano '01; Baswana, Hariharan, Sen '02]
- Deterministic version: [Evald, Fredslund-Hansen, Probst Gutenberg, Wulff-Nilsen '21]

Regime 2: $1 + \epsilon$

Decremental: Pretty well explored

- Total upate time $\tilde{O}(mn)$ [Bernstein '13]
- Deterministic total update time $O(mn^{1+o(1)})$ in undirected graphs [Bernstein, Probst Gutenberg, Saranurak]

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Fully dynamic: [v.d. Brand, Nanongkai '19]

- Update time $\tilde{O}(n^{2.045})$
- Update time $\tilde{O}(n^2)$ in unweighted, undirected graphs
- Trade-offs:
  - Update time $n^{1.863}$, query time $O(n^{0.666})$
  - Update time $n^{1.823}$, query time $O(n^{0.45})$ in unweighted, undirected graphs

Our Results

[v.d. Brand, **F**, Nazari '22]

- Deterministic $1 + \epsilon$ in unweighted, undirected graphs
  - k sources: worst-case update time $O(n^{1.529} + kn^{1+o(1)})$
  - single pair: worst-case update time $O(n^{1.407})$

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Remarks:

- Randomized $O(n^{1.529}+kn^{1+o(1)})$ by [Bergamaschi et al. '21]
- $n^{1.529}$ and $n^{1.407}$ match CLBs [v.d. Brand, Nanongkai, Saranurak '19]

Technique

Two Worlds

- **Combinatorial** emulator with multiplicative stretch $1 + \epsilon$ and additive stretch 4
- **Algebraic** data structure for maintaining small distances

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2. Hitting set for neighborhoods can be maintained with a lazy approach giving low recourse
3. Algebraic data structure can be extended to slowly changing set of nodes

Regime 3: Small constant (2 to < 3)

From now on: all-pairs distances, undirected graphs, amortized update times

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Prior work:

- Fully dynamic: $2 + \epsilon$ with update time $m^{1+o(1)}$ [Bernstein '09]
- Decremental: 2 with total update time $n^{2.5}$ [Henzinger, **K**, Nanongkai '13]

Our Result

[Dory, **F**, Nazari, de Vos arXiv '22]

- Decremental $2+\epsilon$-approximation in weighted graphs
- Total update time $\tilde{O}(m^{1/2}n^{3/2+o(1)})$ if $m \leq n^{5/3}$
- Total update time $\tilde{O}(mn^{2/3})$ if $m \geq n^{5/3}$
- Constant query time



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Main ideas:

- Setup based on bunches and clusters (similar to TZ distance oracle)
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My impression

Static algorithms are still too "messy". A candidate for SOSA?

Regime 4: Thorup-Zwick style (3 to $O(\log n)$)

Partially dynamic:

- Decremental with stretch $(2k-1)(1+\epsilon)$, update time $\tilde{O}((m+n^{1+o(1)})n^{1/k})$, and query time $\min(O(\log\log(nW), k)$ [Chechik '18; Łącki, Nazari '22]
- Incremental with stretch $(2k-1)^t$ and worst case update/query time $\tilde{O}(m^{1/(t+1)}n^{t/k})$ [Chen, Goranci, Monika Henzinger, Peng, Saranurak '20]

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Fully dynamic:

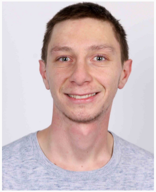
- Stretch $2^{O(\rho k)}$, update time $\tilde{O}(m^{1/2}n^{1/k})$, and query time $O(k^2\rho^2)$ ($\rho = 1 + \lceil \log n^{1-1/k} / \log(m/n^{1-1/k})\rceil$) [Abraham, Chechik, Talwar '14]
- Stretch $\tilde{O}(\log n)$, update/query time $O(m^{2/3+o(1)})$ [Chen, Goranci, Monika Henzinger, Peng, Saranurak '20]

Our Result

[**F**, Goranci, Nazari, Skarlatos '23]

Fully dynamic distance oracle, for any constant $0 < \rho < 1$:

- Stretch $(256/\rho^2)^{4/\rho}$, update time n^ρ, query time $n^{\rho/8}$ or
- Stretch $O(\log\log n)$, update time n^ρ, query time $n^{o(1)}$



Main Idea

Very general reduction

- **Decremental** algorithm maintaining hub set $H(v)$ for every node v such that for any pair s, t:

$$\delta(s,t) = \min_{v \in H(s) \cap H(t)} \delta_H(s,v) + \delta_H(s,t)$$

- can be extended to **fully dynamic** distance oracle

Thorup-Zwick distance oracle as hub labeling scheme

- Stretch $2k-1$
- Hub set of size $O(n^{1/k})$
- *Additional* query mechanism for query time $O(k)$ instead of $O(n^{1/k})$

Independent result [Chuzhoy, Zhang '23]: stretch $(\log\log n)^{2^{1/\rho^3}}$, update time $\tilde{O}(n^{O(\rho)})$, and query time $\tilde{O}(2^{O(1/\rho)})$ for any $\frac{2}{(\log n)^{1/200}} < \rho < \frac{1}{400}$

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- Polylogarithmic stretch, total update time $m^{1+\rho}$ for any constant ρ
- Stretch $n^{o(1)}$, total update time $m^{1+o(1)}$

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Fully dynamic:

- Stretch $O(\log n)^{3k-2}$, update time time $m^{1/k+o(1)} \cdot O(\log n)^{4k-2}$, and query time $O(k(\log n)^2)$ [Forster, Goranci, Henzinger '21]

Our Result

[**F**, Nazari, Probst Gutenberg '23]

- **Incremental**
- Polylogarithmic stretch
- Total update time $\tilde{O}(m)$
- Query time $O(\log \log n)$

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Notable:

- No Even-Shiloach tree
- No expander decomposition
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Overall setup:

- Hierarchy of $k = \Theta(\log \log n)$ sparsifiers: $G = H_1, H_2, \ldots, H_k$ following [Andoni, Stein, Zhong '20]
- H_i is an α-approximation of H_{i-1} for some constant α
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Challenge:

- Naively, number of edges added to H_{i+1}: $O(\ldots + |E(H_i)| \log n)$
- For $k = \Theta(\log \log n)$ levels: $m \cdot O(\log n)^{\log \log n}$ insertions (at best)

Open Problems

I would like to see the following:

1. Conditional lower bound of $n^{2.5}$ for worst-case update time in exact APSP
2. Stretch $(2k-1)$, (worst case) update time $O(n^{1/k})$, query time $O(k)$
3. Deterministic decremental with polylogarithmic stretch, polylogarithmic query time and update time $\tilde{O}(m)$