Dynamic algorithms for k-center on graphs

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@Simons Institute (September 19, 2023)

Joint work with Emilio Cruciani, Gramoz Goranci, Yasamin Nazari, and Antonis Skarlatos

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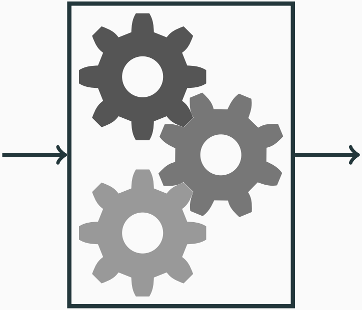
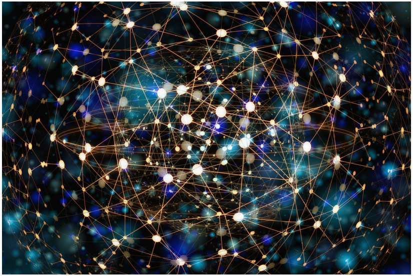
Dynamic Algorithms

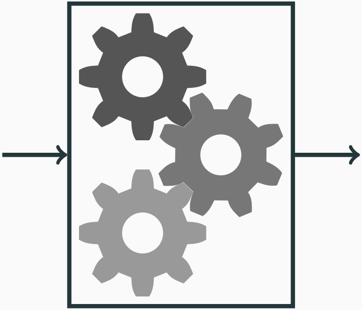


Network Science



Dynamic Graph Clustering

Dynamic Algorithms



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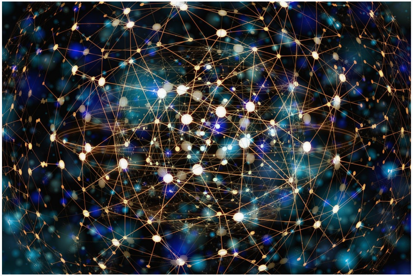
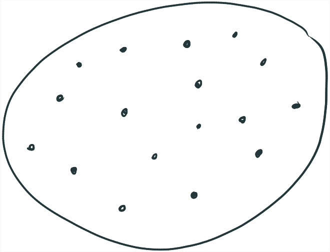


Prior algorithms on dynamic clustering not tailored to graphs!

k-Center Problem

Given a metric space, select k points as set of centers C such that the maximum distance $d(C, v)$ of any node v to its closest center is minimized.



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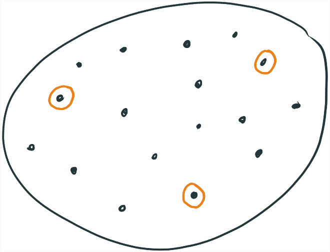

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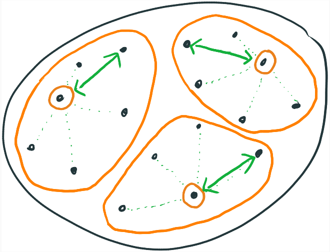
- Radius of a cluster: Maximum distance of the center to the nodes in its cluster



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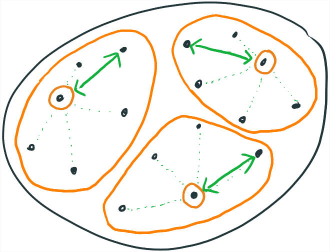
- Assigning each point to its closest center induces a partition into clusters
- Radius of a cluster: Maximum distance of the center to the nodes in its cluster
- Problem is NP-hard to approximate within a factor of $2 - \epsilon$



Metric Spaces and Graphs

Definition (Metric on Point Set)

1. **Non-Negativity:** $d(x, y) \geq 0$
2. **Separation:** $d(x, y) = 0$ if and only if $x = y$
3. **Symmetry:** $d(x, y) = d(y, x)$
4. **Triangle inequality:** $d(x, z) \leq d(x, y) + d(y, z)$

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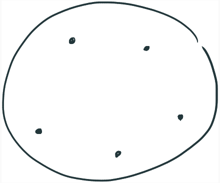
Question

Are there efficient dynamic constant-factor approximation algorithms for k-center if the metric is induced by a dynamically changing undirected graph?

Dynamic Model

Dynamic point sets:

- Point insertions and deletions



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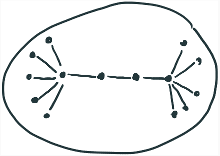

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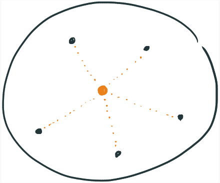

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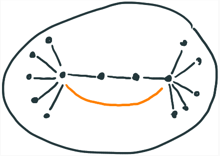
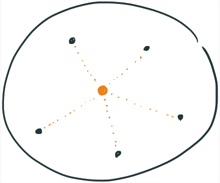


Dynamic Model

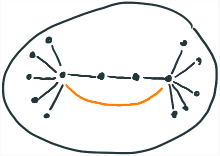
Dynamic point sets:

- Point insertions and deletions
- Query access to metric



Dynamic graphs:

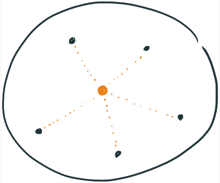
- Edge insertions and deletions
- Distances not given for free



Dynamic Model

Dynamic point sets:

- Point insertions and deletions
- Query access to metric
- Metric is extended/reduced



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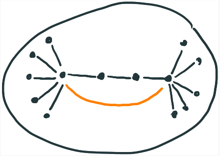

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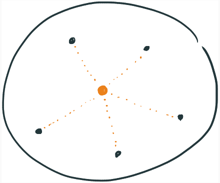

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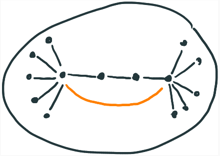

Conclusion

Cannot use results for dynamic point sets in a black-box manner for dynamic graph model

Related Work

Static algorithms:

- Classic 2-approximation algorithms [Gonzalez '85] [Hochbaum, Shmoys '85]
  On graphs with n nodes and m edges: $\tilde{O}(km)$ time
- State of the art on graphs: $\tilde{O}(m)$ time (randomized) [Thorup '01] [Abboud et al. '23]

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Dynamic point sets:

- $\tilde{O}(k^2)$ update time [Chan, Gourqin, Sozio '18]
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- Special cases: [Schmidt, Sohler '19] [Goranci et al. '21]
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Natural goal: Update-time overhead of $\tilde{O}(k)$ compared to dynamic approximate SSSP

Our Results I: Fully Dynamic

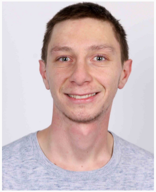
Theorem (Cruciani, F, Goranci, Nazari, Skarlatos ’23)

There is a fully dynamic $(2+\epsilon)$-approximate k-center algorithm with worst-case update time

- $O(kn^{1.529}\epsilon^{-2})$ *in unweighted graphs*
- $O(kn^{1.823}\epsilon^{-2})$ *in weighted graphs*

that is correct against an adaptive adversary.



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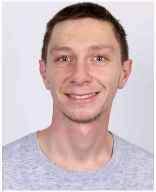
Update time for fully dynamic $(1+\epsilon)$-approximate SSSP:

- $O(n^{1.529}\epsilon^{-2})$ (unweighted) [v. d. Brand, **F**, Nazari ’22]
- $O(n^{1.823}\epsilon^{-2})$ (weighted) [v. d. Brand, Nanongkai ’19]



Our Results II: Partially Dynamic

Theorem (Cruciani, F, Goranci, Nazari, Skarlatos '23)

There is a deterministic decremental $(2+\epsilon)$-approximate k-center algorithm with amortized update time $kn^{o(1)}$ (over a sequence of $\Theta(m)$ updates).

(in this talk: constant ϵ, polynomially bounded integer edge weights)

Update time for decremental $(1+\epsilon)$-approximate SSSP: $n^{o(1)}$

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Update time for decremental $(1+\epsilon)$-approximate SSSP: $n^{o(1)}$

Theorem (Cruciani, F, Goranci, Nazari, Skarlatos '23)

There is a randomized incremental $(4+\epsilon)$-approximate k-center algorithm with amortized update time $kn^{o(1)}$ that is correct against an oblivious adversary.

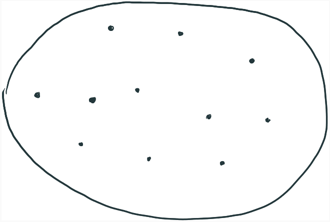
Update time for incremental $(1+\epsilon)$-approximate SSSP: $n^{o(1)}$

Warm-Up: Fully Dynamic Algorithm

Reminder: Gonzalez's Algorithm

Gonzalez's Algorithm [Gonzalez '85]

1. Initialize $C = \{v\}$ with arbitrary first center
2. While $|C| < k$, add node v maximizing $d(C, v)$ to C



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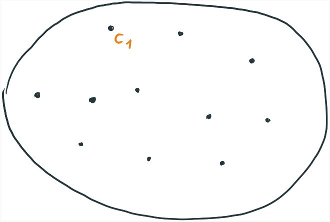

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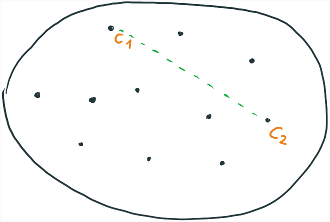

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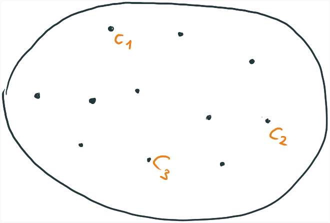

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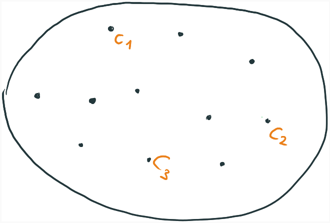

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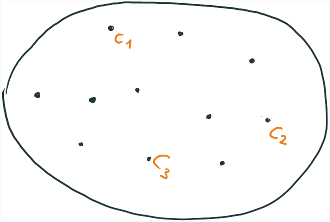

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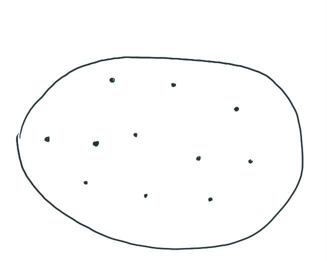
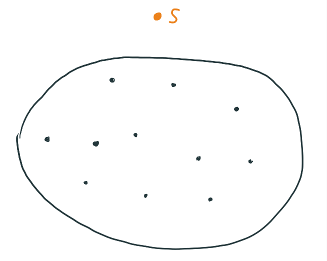
If $d(C, v)$ is within factor $1 + \epsilon$ of maximum, this gives $(2 + \epsilon)$-approximation

Reduction to SSSP: Simulating Gonzalez's Algorithm



Reduction to SSSP: Simulating Gonzalez's Algorithm

- Add artificial "super-source" s
- Maintain $(1+\epsilon)$-approximate single-source distances from s with a fully dynamic algorithm



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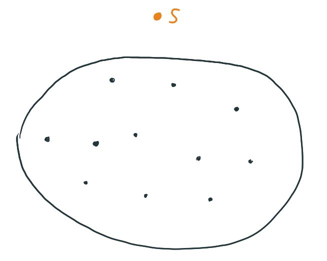

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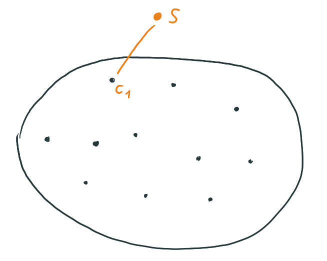

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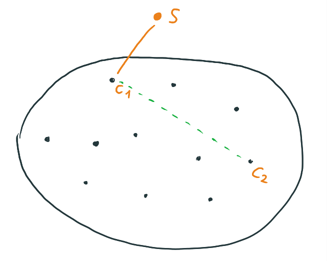

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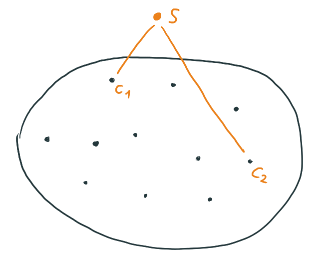

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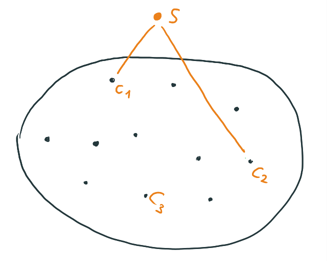

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Reduction to SSSP: Simulating Gonzalez's Algorithm

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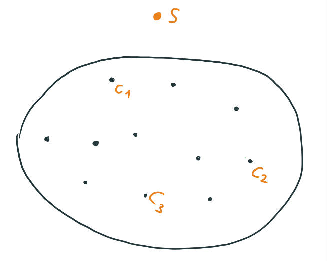

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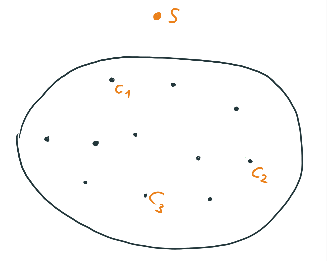

Update Time: $O(k \cdot U_{\mathrm{SSSP}}(n))$

Partially Dynamic Algorithms

R-Independent Sets

Definition (R-Independent Set)

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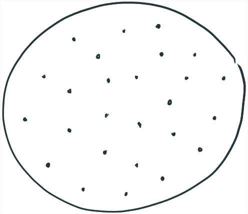

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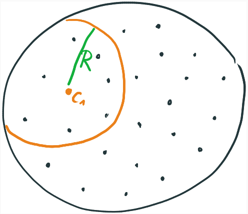

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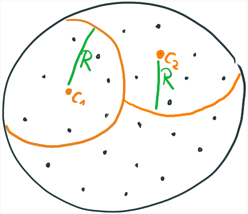

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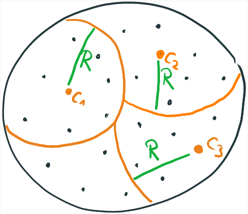

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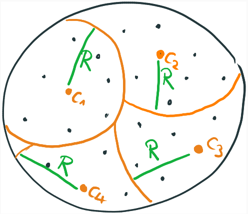

Definition (R-Independent Set)

- $C \subseteq V$ such that $d(u, v) > R$ for all $u, v \in C$
- *maximal* if C cannot be extended without violating the property

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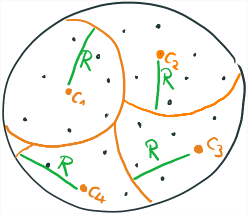

"k-bounded maximal R-Independent set C":

- $|C| < k$ and $|C|$ is maximal or
- $|C| = k$

k-Center Based on R-Independent Sets [Hochbaum, Shmoys '85]

Goal: Find smallest value of R such that maximal R-independent set has size $\leq k$

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Again: $(1 + \epsilon)$-approximate distances lead to $(2 + \epsilon)$-approximation

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Observations

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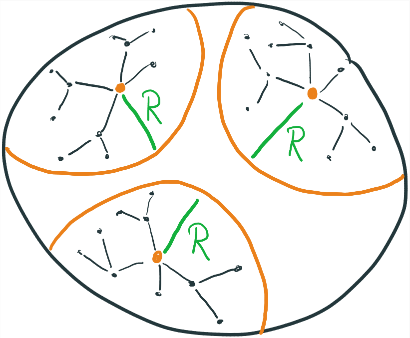
- After each deletion:
  - Forward update to SSSP data structure
  - If there is a node with $d(C, v) > (1 + \epsilon)R$:
    - If $|C| = k$, then $R \leftarrow (1 + \epsilon)R$
    - Otherwise: Add v to C and restart decremental SSSP

Example and Running Time



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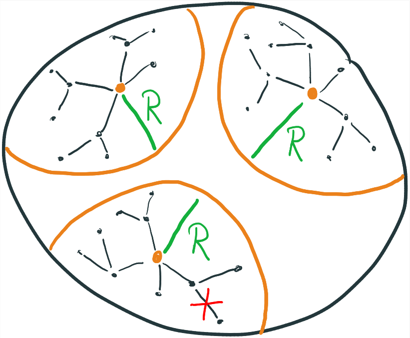

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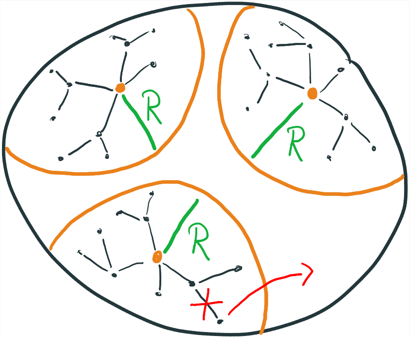

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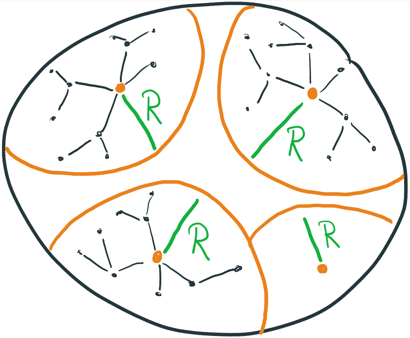

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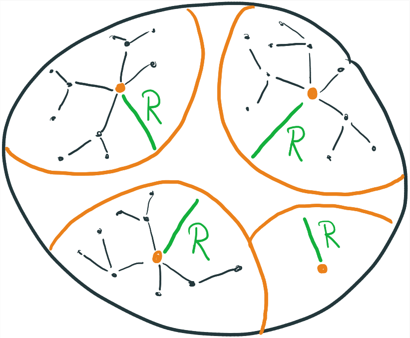

Total update time: $O(k \log_{1+\epsilon}(n\Lambda)) \times T_{\mathrm{SSSP}}(m)$

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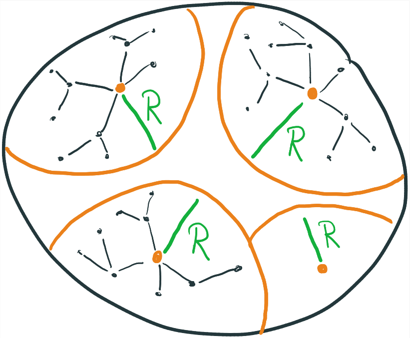

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- $T_{\mathrm{SSSP}}(m) = m^{1+o(1)}$ [Henzinger, **K**, Nanongkai '14], also deterministically [Bernstein, Probst Gutenberg, Saranurak '21]

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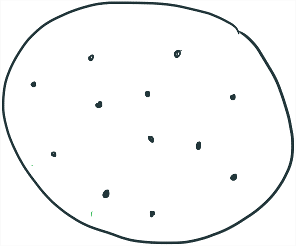
- **Goal:** Maintain R-independent sets with low **total** recourse

MIS Abstraction [Bateni et al. '23]

R-**neighborhood graph** G_R: edge (x, y) if $d(x, y) \leq R$



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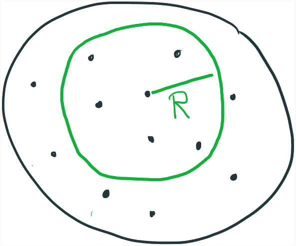

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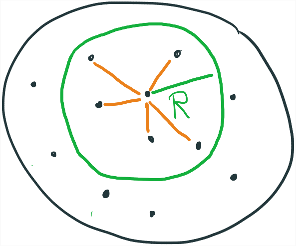

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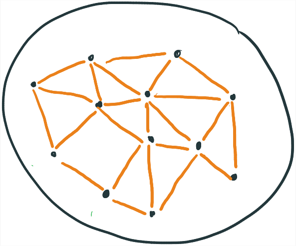

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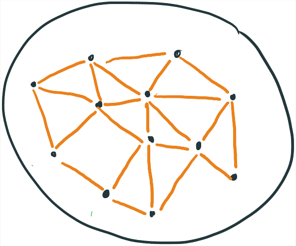

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R-independent set in original graph

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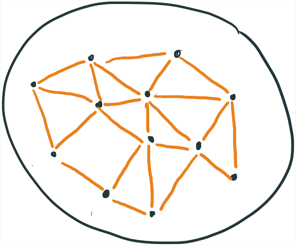

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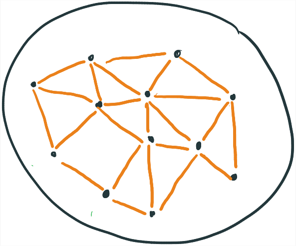

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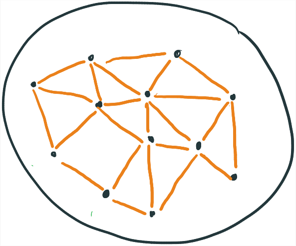

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- Neighborhood graph essentially an analysis tool, only constructed partially
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- Clean formal definition for algorithmically defined approximate distances is tricky (but also not necessary)

Dominating Sets to the Rescue

Goal:

- Maintain k-bounded MIS in neighborhood graph G_R with total recourse poly(k)

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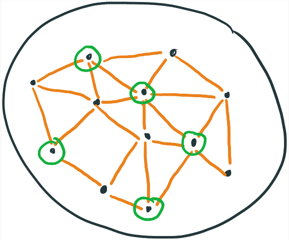

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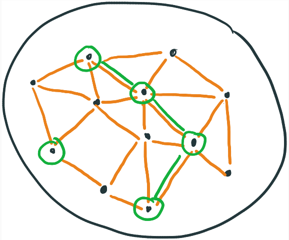

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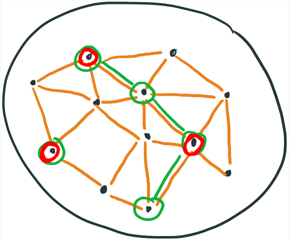

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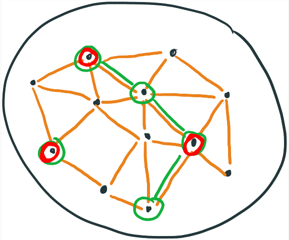
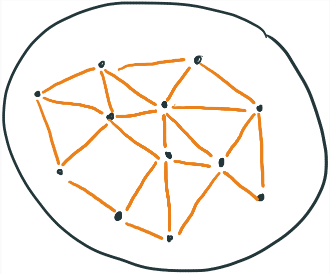
- Every node at distance ≤ 2 to a center in G_R, and thus at distance $\leq 2R$ in G $\rightarrow (4+\epsilon)$-approximation



Dominating Sets via Random Sampling

Static sampling algorithm:



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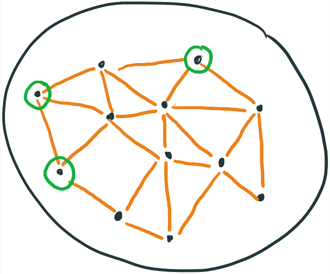

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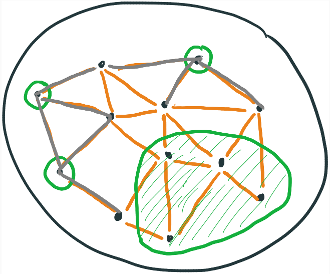

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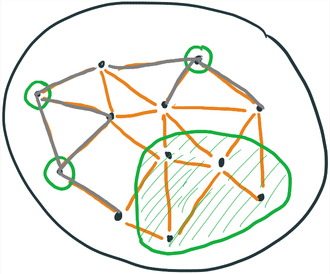

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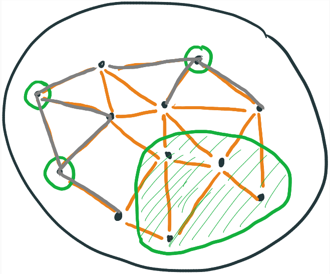

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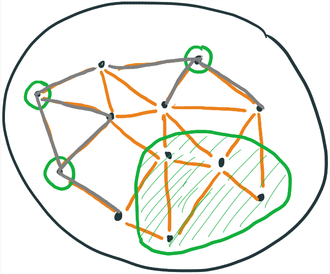
→ Algorithm only needs access to edges incident on S in G_R

→ Incremental algorithm very similar



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Consequence:

- $O(\log n)$ iterations of sampling procedure
- S has $\tilde{O}(k)$ nodes

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Question

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- Attention: Recourse guarantee is needed for dense neighborhood graph

Conclusion

Remarks

Nice application of dynamic approximate SSSP

Path-reporting seems to be less relevant in this context

Incremental was the difficult question for this problem

Incremental model more relevant than we usually consider

Interesting question about dynamic MIS suddenly shows up

Future Work

Consider other clustering objectives in graphs

Engineer dynamic approximate SSSP algorithm

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Thank you!